Two Alternatives to the Shewhart $\bar{X}$ Control Chart

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Average run length (ARL) values are calculated for two $\bar{X}$ control chart schemes and compared with those of a standard Shewhart chart. Both control charts are based on runs rules and are easily implemented. An out-of-control condition for one of the charts is a run of two of two successive points beyond a special control limit. The other chart uses a run of two of three successive points beyond a different control limit. Both schemes are shown to have better, that is, lower, ARL values than the standard Shewhart chart for process average shifts as large as 2.6 standard deviations from the mean.

Introduction

The Shewhart $\bar{X}$ control chart, which signals an out-of-control condition when a single point falls beyond a three-sigma limit, has been the standard control chart for variables and attributes since the first quarter of this century. Its attractiveness is rooted in its simplicity and its ability to detect large process average shifts quickly. However, it is also known to be relatively insensitive to small-sustained process average shifts.

The first proposal (Shewhart (1941)) to make the Shewhart chart more sensitive to small shifts in the process average suggested additional tests in the form of runs rules. Three such tests are described in Western Electric (1956), Nelson (1984), and Montgomery (1997). They describe an out-of-control condition if k of n successive points fall beyond one-, two-, or three-sigma limits, where $2 \leq k \leq n$. While these simultaneous tests achieve their goal, they do so at the expense of significant increases in false out-of-control signals, as shown in Champ and Woodall's (1987) important study. Indeed, Montgomery (1997) suggested that the additional sensitizing rules should be used cautiously, because of the potentially deleterious effects of false alarms.

Alternative control chart methodologies have been suggested, for example, the cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) schemes. Both of these have excellent small process average shift detection capabilities, as described in Montgomery (1997). However, so far, they do not seem to have achieved widespread application beyond the chemical process industries. This may be due to a perception that the required calculations are too complex for typical shop floor work and/or to the usual organizational inertia associated with procedural changes. Therefore, in this paper, we consider variations of simpler traditional methods, which may be more acceptable to practitioners.

For pedagogical purposes, Derman and Ross (1997) considered two additional schemes, each of which used specially designated (lower than three-sigma) control limits. In the first, an out-of-control signal is given if two successive points fall outside either of such control limits. In the second, an out-of-control signal is obtained if any two of three successive points fall beyond different (less than three-sigma) control limits. Thus, in their first scheme, given two successive points, an out of control signal is obtained if either point is above an upper control limit and the other is below a lower control limit, or if both points are beyond one limit. In their second scheme, an out-of-control signal is obtained if any two of three successive points are above, or below, either control limit. In brief, they showed that that both schemes provided increased sensitivity to moderate process average shifts over that of a standard Shewhart control chart. Detailed comparisons will be given in the next section.

Here, we consider two slightly different alternatives to the Shewhart $\bar{X}$ chart. In the first, either

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two successive points above an upper control limit or two successive points below a lower control limit are needed to obtain an out-of-control signal. In the second, the out-of control signal is given if two of three successive points are above an upper control limit or two of three are below a lower control limit. The choice of such limits is discussed in the appendices.

Note that a "two-of-two" rule is the simplest runs-rule requiring more than one observation to obtain an out-of-control signal. Also, the "two-of-three" rule is a standard runs-rule, except that the upper and lower limits suggested here are different than the usual "two-sigma" lines. Equally important for the shop floor, both of these schemes are simple and easy to implement. It will be shown that both are more sensitive to process average changes than the standard Shewhart control chart (with no supplementary tests) over an important range of such changes.

We evaluate all of the schemes by comparing their average run length (ARL) profiles with that of the Shewhart scheme. Recall that for a specified control chart scheme and a given process average shift, an ARL is the expected number of points plotted on the chart until an out-of-control signal is obtained. An ARL profile is the sequence of ARL values associated with a sequence of process average shifts. The ARL value associated with a zero-valued shift is called the "in-control" value. It represents the expected number of points until a false out-of-control signal is obtained.

For all calculations, we assume that the random variables giving rise to the points plotted on a control chart are independent and normal with a standard deviation equal to one. The mean of the distribution is zero when the process is in-control. The process is considered to be out-of-control when the process average is non-zero. Because the normal distribution is symmetric about it's mean, the ARL profile values are the same for equal positive and negative process average shifts. Changes in the process average, measured in standard deviation units, are assumed to be sudden and sustained.

In brief, our results show that both of our alternative control schemes have better ARL profiles than the standard Shewhart scheme for process average shifts up to about 2.6σ. For larger process average shifts, the Shewhart chart ARL profile is marginally better. However, in many applications, either of our two alternative schemes should be more useful than the Shewhart chart, when the latter is used without adjunct runs-rules.

Results

Table 1 contains ARL values for a standard Shewhart three-sigma $\overline{X}$ control chart for process average shifts from zero (in-control) to out-of-control values

TABLE 1. ARL Profiles

	Shewhart	D-R:2 of 2	D-R:2 of 3	2 of 2	2 of 3
	Control Limits				
Shift	±3	±1.9322	±2.0698	±1.7814	±1.9307
0	370	370	370	**370**	**370**
0.2	308	313	308	**277**	**271**
0.4	200	204	193	**150**	**142**
0.6	120	116	107	**79**	**73**
0.8	72	65	58	**44**	**40**
1	44	37	33	**26**	**23**
1.2	28	23	20	**16**	**15**
1.4	18	15	13	**11**	**10**
1.6	12	10	8.9	**7.8**	**7.1**
1.8	8.7	7.2	6.6	**5.9**	**5.4**
2	6.3	5.5	5.1	**4.6**	**4.3**
2.2	4.7	4.4	4.1	**3.8**	**3.6**
2.4	3.6	3.6	3.4	**3.2**	**3.1**
2.6	2.9	3.1	3.0	**2.8**	**2.8**
2.8	**2.4**	2.8	2.7	2.6	2.5
3	**2.0**	2.5	2.5	2.4	2.4
4	**1.2**	2.1	2.1	2.0	2.0
5	**1.0**	2.0	2.0	2.0	2.0

up to five-sigma. These ARL values are adjacent to comparable values for the Derman-Ross (D-R) and our two-of-two and the two-of-three runs-rules schemes. The parameters used for each of these schemes are also listed. The method used to determine the values of the parameters is described in the appendices. For process average shifts from 0 to 2.6 standard deviations, we see that the ARL values for both of our suggested rules (boldfaced) are lower than those of either the Shewhart or the D-R schemes. In addition, for process average shifts greater than 2.6, the ARL profile advantages of the Shewhart scheme are, at most equal to one, and for many practical applications, negligible.

Although we have not included ARL profiles for our rules, using respectively, rounded control limits of 1.78 and 1.93, their values are essentially the same as those shown in Table 1.

Concluding Remarks

In addition to its favorable ARL profile, the two-of-three rule may be attractive to practitioners who have been using such a rule as an adjunct to the standard Shewhart control chart. However, despite its slight advantage over the two-of-two rule, we find the simplicity of the latter more appealing.

We note that our proposed control charts, in ARL terms, are not as good as EWMA schemes. For example, Saccucci and Lucas (1990) gave some ARL profile values for an EWMA scheme with parameters $L = 2.9$ and $\lambda = 0.25$. This scheme has an in-control ARL value of 370 and ARL values of 41, 10 and 3.5 for process average shifts of 0.5, 1.0 and 2.0. These ARL values dominate those of our "two of three" control chart of 101, 23 and 4.3 for the same process average shifts.

To the best of our knowledge, a two of two rule was first used by Hurwitz and Mathur (1992). They used it to replace all three standard runs-rules in an industrial setting because of operational difficulties experienced in using three rules as adjuncts to the standard Shewhart chart. They used two of two control limits of 1.5σ, to approximately match the in-control ARL (92) of the Shewhart chart with three adjunct runs-rules. It is interesting that the relatively high incidence of false out-of-control signals, obtained from the original and from this replacement scheme, was not reported as troublesome.

A two of two runs-rule was also used, by Klein (1997), in the development and evaluation of some Shewhart-EWMA $\overline{X}$ control chart schemes. In that study, it was used as a device to reduce the variance of the distribution of the in-control run length.

Appendix A: Computational Details For the Two of Two Scheme

The ARL profile for this scheme is to be compared with that of a standard Shewhart chart. Hence, appropriate upper and lower control limits have to be found so that the in-control ARL value for this scheme will be equal to 370.4, the same value as that of the *standard* Shewhart control chart.

A control chart can be viewed as consisting of three regions: one above the upper control limit, one below the lower control limit and a center region between the two limits. We denote the probability of a single point falling in the upper region by pU, in the lower region by pL, and in the center region by p. The values of these probabilities determine the location of the control limits.

Consider an absorbing Markov chain with three transient states $\{1, 2, 3\}$: state $\{1\}$ representing no points beyond either control limit, state $\{2\}$ a point above the upper control limit (UCL), and state $\{3\}$ a point below the lower control limit (LCL). The process reaches the absorbing state $\{4\}$ when two successive points are beyond just one of the control limits.

The transition probabilities of the chain are given in Table A1, with transition probabilities pU, pL and p as defined above. The expected value of the first passage time from starting state 1 to the absorbing (out-of-control) state 4 is equal to the in-control average run length. The expected first passage times (number of transitions) from each of the states to the absorbing state can be determined by solving the linear system given below, see, for example, Derman, Gelser, and Olkin (1973):

TABLE A1. Transition Probabilities for Markov Chain With 3 Transient States

		States at time $t+1$			
		1	2	3	4
States at time t	1	p	pU	pL	0
	2	p	0	pL	pU
	3	p	pU	0	pL
	4	0	0	0	1

$$M_{14} = 1 + (p)M_{14} + (pU)M_{24} + (pL)M_{34}$$
$$M_{24} = 1 + (p)M_{14} + (pL)M_{34}$$
$$M_{34} = 1 + (p)M_{14} + (pU)M_{24}\,.$$

Here, M_{14} is the expected first passage time from the starting state $\{1\}$ to the out-of-control, absorbing, state $\{4\}$, that is, it is the average in-control run length. One form of the solution, of this system, is the following formula, due to Hurwitz and Mathur (1992) for the average run length:

$$M_{14} = \frac{1}{1 - p - \frac{pU}{1+pU} - \frac{pL}{1+pL}}\,. \qquad \text{(A1)}$$

Since we use symmetrical control limits, setting $pL = pU = p^*$ and noting that $1 - p = pL + pU = 2p^*$, it is easy to show that the above reduces to

$$M_{14} = \frac{1+p^*}{2p^*}\,.$$

Now, set $M_{14} = \text{ARL} = 370.4$ to match the in-control ARL of the *standard* Shewhart scheme, and solve for p^* to obtain $p^* = pL = pU = 0.037422$. Given, $pU = pL = 0.037422$, using standard Normal tables, the lower control limit is -1.7814 and the upper control limit is $+1.7814$.

For process average shift values of, say, $b > 0$, the calculation of the out-of-control average run lengths requires the recalculation of the probabilities pL and pU, modified to account for the shift in the normal distribution relative to the above control limits. Then Equation (A1) is used to determine the ARL values associated with the shift. For example, for a process average shift of b standard deviations, we have

$$pL = \Pr\{X + b < -1.7814\}$$
$$= \Pr\{X < -1.7814 - b\} \qquad \text{(A2)}$$

where $X \sim N(0,1)$. Thus, if $b = 1$, we have $pL = \Pr\{X < -2.7814\} = 0.002706$. Similarly, $pU = \Pr\{X > 0.7814\} = 0.217278$, and using, Equation (A1), the ARL, for $b = 1$, is 25.78.

Appendix B: Computational Details For the Two of Three Scheme

The approach is the same as that used above except that the absorbing Markov chain increases in size to eight states $\{1, 2, \ldots, 8\}$ with the first seven of them as transient (see Table B1). The states are:

State (OO) has two successive points between both control limits;

State (OU) has a first point between both control limits and the second above the UCL;

State (OL) has a first point between both control limits and the second below the LCL;

State (UL) has its first point above the UCL and its second below the LCL;

State (UO) has its first point above the UCL and its second between the control limits;

State (LO) has its first point below the LCL and its second between the control limits;

State (LU) has its first point below the LCL and its second above the UCL;

State (OOC) the absorbing state, has two of three points either below the LCL or above the UCL.

As in our earlier calculation, the expected first passage times (number of transitions) from each of the the absorbing state can be determined by states to solving the linear system given below. The expected

TABLE B1. Transition Probabilities for Markov Chain With 7 Transient States

			States at time $t+1$							
			1	2	3	4	5	6	7	8
			(OO)	(OU)	(OL)	(UL)	(UO)	(LO)	(LU)	(OOC)
States at time t	1	(OO)	p	pU	pL					
	2	(OU)				pL	p			pU
	3	(OL)						p	pU	pL
	4	(UL)						p		$pL + pU$
	5	(UO)	p		pL					pU
	6	(LO)	p	pU						pL
	7	(LU)					p			$pL + pU$
	8	(OOC)								1

value of the first passage time from starting state 1 to the absorbing (out-of-control) state 8 is equal to the in-control average run length.

$$\begin{aligned} M_{18} &= 1 + (p)M_{18} + (pU)M_{28} + (pL)M_{38} \\ M_{28} &= 1 + (p)M_{58} + (pL)M_{48} \\ M_{38} &= 1 + (p)M_{68} + (pU)M_{78} \\ M_{48} &= 1 + (p)M_{68} \\ M_{58} &= 1 + (p)M_{18} + (pL)M_{38} \\ M_{68} &= 1 + (p)M_{18} + (pU)M_{28} \\ M_{78} &= 1 + (p)M_{58}\,. \end{aligned} \tag{B1}$$

Here, M_{18} is the average in-control run length.

To find the control limits to obtain an in-control ARL = 370.4, we solve the above system with the additional constraints: $pL = pU, pL + pU + p = 1$, $M_{18} = 370.4$. The solution of this system is $pU = pL = 0.02676$. From standard normal tables, we find that the control limits are ± 1.9307. New values of pL and pU resulting from process average shifts are found using standard normal tables and calculations using Equation (A2), etc. Then, we use the new values of pL and pU to find the ARL associated with the process average shift specified, by solving the above system of equations in Equation (B1). For example for a process average shift of $b = 1$ and a lower control limit of -1.9307, we have $pL = \Pr\{X + 1 < -1.9307\}$, where $X \sim N(0, 1)$. Hence, $pL = 0.001691$. Similarly, $pU = \Pr\{X > 0.9307\} = 0.176004$ and, solving the equations in Equation (B1), the ARL, for $b = 1$, is 23.3747.

All ARL profile calculations were done using Microsoft Excel. The Derman-Ross schemes evaluations were done by similar methods, except that we used the explicit formulas for the ARL calculations given in Derman and Ross (1997).

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