Optimal design of a maintainable cold-standby system

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Introduction

- The availability is often required for a maintainable system :

$$A_S \geq A_0$$

- High level of the system performance often indicates high system cost, rendering optimal problems.
- Under a reliability constraint, the optimal object is to minimize the construction cost of the system, and with an availability constraint, often is minimized the system cost rate.

Cold-standby system

- A method of redundancy in which the secondary system is only called upon when the primary system fails. It is used for non-critical applications or in cases where data is changed infrequently.

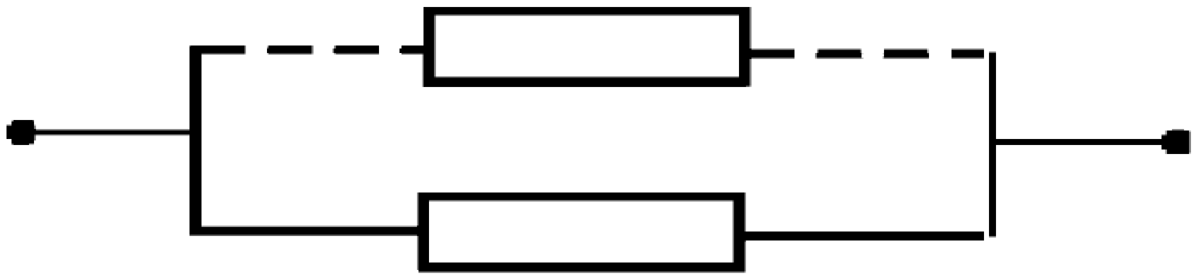

Fig. 1. A two-component cold-standby system.

Problem description

- Our goal is to minimize the long-run cost rate α_S of the system, by determining the "best" component and the "best" maintenance service.

$$\min_{t_M \geqslant 0, \mu_F > 0} \alpha_S(t_M, \mu_F) \quad \text{subject to : } A_S(t_M, \mu_F) \geqslant A_0.$$

Nomenclature

A_0	system availability requirement
A_S	system availability
t_F	time to failure of the component
t_M	recovery time of the maintenance
μ_F	mean time to failure of the component
α_S	long-run cost rate of the system
$f(t)$	failure distribution of the component
$P(\cdot)$	probability function
$E(\cdot)$	expectation function
n	times of maintenances
H_n	time to the termination of the nth maintenance
i	index of the working components
j	index of the uncompleted maintenances before the operating component fails

C_M	cost of the maintenance
C_U	cost of the system unavailability
$C_S(H_n)$	system cumulative cost during the period $[0, H_n]$
$n_{t_F<t_R}$	times of $t_F < t_M$ within n times of maintenances
$\lfloor \cdot \rfloor$	function of rounding towards 0
$N_{t_F<t_R}$	round-down integer of the expected value of $n_{t_F<t_R}$
ξ_{C_U}	conditional expected value of C_U
p	probability that $t_F < t_M$
q	conditional expected value of $(t_M - t_F)$ when $t_F < t_M$
D	set of available types of components
T	upper bound of t_M
θ, β	scale parameter and shape parameter of Weibull distribution

Probability analysis-1

$$p \equiv P(t_F < t_M) = \int_0^{t_M} f(t_F)\,\mathrm{d}t_F$$

$$q \equiv E(t_M - t_F | t_F < t_M) = \frac{\int_0^{t_M} (t_M - t_F) f(t_F)\,\mathrm{d}t_F}{\int_0^{t_M} f(t_F)\,\mathrm{d}t_F}.$$

$$C_M = C_M(t_M, \mu_F), \quad C_U = C_U(t_M - t_F | t_F < t_M).$$

$$H_n = \sum_{i=1}^{n} t_F^i + \sum_{j=1}^{n_{t_F < t_M}} \left(t_M - t_F^j\right)$$

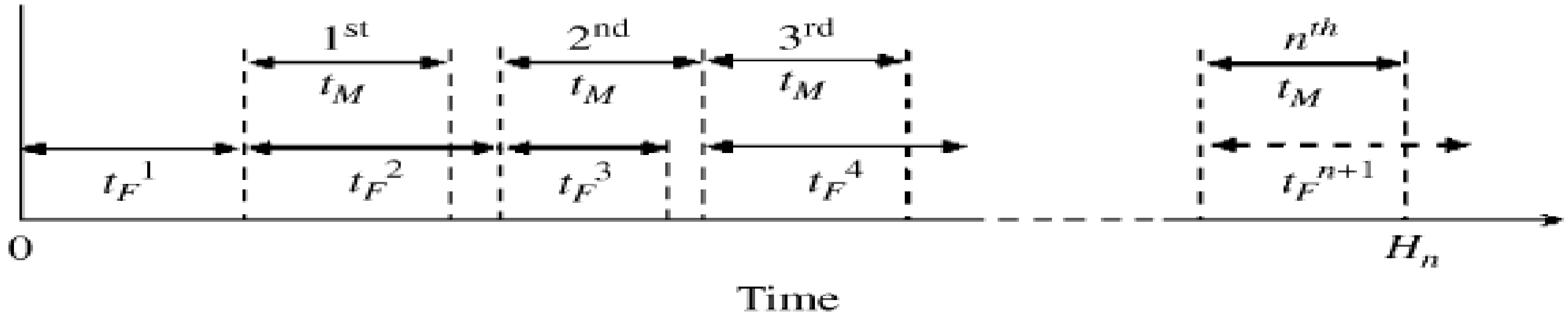

Fig. 2. Calculation of the time to the termination of the nth maintenance.

Probability analysis-2

$$C_S(H_n) = nC_M(t_M, \mu_F) + \sum_{j=1}^{n_{t_F < t_M}} C_U\left(t_M - t_F^j\right).$$

$$\alpha_S(t_M, \mu_F) = \lim_{H_n \to \infty} \frac{C_S(H_n)}{H_n}$$

$$= \lim_{n \to \infty} \frac{nC_M(t_M, \mu_F) + \sum_{j=1}^{n_{t_F < t_M}} C_U\left(t_M - t_F^j\right)}{\sum_{i=1}^{n} t_F^i + \sum_{j=1}^{n_{t_F < t_M}} \left(t_M - t_F^j\right)}$$

$$= \frac{C_M(t_M, \mu_F) + P(t_F < t_M) \cdot E[C_U(t_M - t_F) | t_F < t_M]}{\mu_F + P(t_F < t_M) \cdot E(t_M - t_F | t_F < t_M)}$$

$$A_S(t_M, \mu_F) = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} t_F^i}{\sum_{i=1}^{n} t_F^i + \sum_{j=1}^{n_{t_F < t_M}} \left(t_M - t_F^j\right)}.$$

$$= \frac{\mu_F}{\mu_F + P(t_F < t_M) E(t_M - t_F | t_F < t_M)}$$

Probability analysis-3

$$\min \alpha_S(t_M, \mu_F) = \frac{C_M + p \cdot \xi c_U}{\mu_F + p \cdot q}$$

$$\text{subject to } A_S(t_M, \mu_F) = \frac{\mu_F}{\mu_F + p \cdot q} \geqslant A_0.$$

Finite set of components

$$f(t) = \frac{a_1 a_2}{a_2 - a_1}(\mathrm{e}^{-a_1 t} - \mathrm{e}^{-a_2 t}), \quad a_2 > a_1 > 0.$$

$$\mu_F = \int_0^\infty t f(t)\,\mathrm{d}t = \frac{a_1 + a_2}{a_1 a_2}$$

$$p = P(t_F < t_M) = 1 - \frac{a_2 \mathrm{e}^{-a_1 t_M} - a_1 \mathrm{e}^{-a_2 t_M}}{a_2 - a_1},$$

$$\begin{aligned} q &= E(t_M - t_F | t_F < t_M) \\ &= \left\{ t_M - \frac{a_2 + a_1}{a_1 a_2}\left[1 - \frac{a_2^2 \mathrm{e}^{-a_1 t_M} - a_1^2 \mathrm{e}^{-a_2 t_M}}{a_2^2 - a_1^2}\right]\right\} \Big/ p(t_M) \end{aligned}$$

$$\begin{aligned} \xi_{C_U} &\equiv E[C_U(t_M - t_F) | t_F < t_M] \\ &= \frac{\int_0^{t_M} C_U(t_M - t) f(t)\,\mathrm{d}t}{p(t_M)}. \end{aligned}$$

- Obviously, μ_F is totally determined by f(t), which is the actual decision variable.

$$f(t) \in D \equiv \{f_l(t), 1 \leqslant l \leqslant m\}.$$

- The problem will be

$$\min_{t_M \geqslant 0, D} \alpha_S = \frac{C_M + p \cdot \xi_{C_U}}{\mu_F + p \cdot q}$$

$$\text{subject to } A_S(t_M, \mu_F) = \frac{\mu_F}{\mu_F + p \cdot q} \geqslant A_0.$$

An underlying property

- Let us focus on the maintenance time t_m.

$$\frac{\mathrm{d}[p(t_M)\cdot q(t_M)]}{\mathrm{d}t_M} = p(t_M) > 0,$$

$$\frac{\mathrm{d}p(t_M)}{\mathrm{d}t_M} = f(t_M) > 0.$$

- This property implies that the availability constraint equals to $0 \le t_m \le T$, where T is the unique solution of $A_S(T, \mu_F) = A_0$.

$$\min_{0 \leqslant t_M \leqslant T, D} \alpha_S = \frac{C_M + p \cdot \xi_{C_U}}{\mu_F + p \cdot q}.$$

Optimal solution

- Step1.Choose one pdf $f_l(t)$ from the given finite set D.
- Step2.Calculate μ_F^l, p^l, q^l and $C_M^l, \xi_{C_U}^l$ to from $\alpha_S = \alpha_S^l(t_M)$
- Step3.Solve the equation $A_S(T^l, \mu_F^l) = A_0$ to get T^l.

- Step4.Find the optimal solution of the sub-problem :

$$\min_{0 \leqslant t_M \leqslant T^l} \alpha_S^l(t_M) = \frac{C_M(t_M)^l + p(t_M)^l \cdot \xi_{C_U}(t_M)^l}{\mu_F^l + p(t_M)^l \cdot q(t_M)^l}.$$

If $\mathrm{d}\alpha_S^l(x)/\mathrm{d}x = 0$ has no solution in $[0, T^l]$, the solution will be

$$^{*}t_M^l = \begin{cases} T^l, & \alpha_S^l(T^l) \leqslant \alpha^l(0), \\ 0, & \alpha_S^l(T^l) > \alpha^l(0). \end{cases}$$

OtherwiseLet

$$B^l = \left\langle x \mid x \in [0, T^l] \& \frac{d\alpha_S^l(x)}{dx} = 0 \right\rangle$$

be the solution set of $\frac{d\alpha_S^l(x)}{dx} = 0$ that in $[0, T^l]$
The optimal solution is $^*t_M^l = x_O$, where $x_O \in B^l$
and $\alpha_S^l(x_O) = \min_{x \in B^l} \alpha^l(x)$.

- Step5.The optimal solution of $A_S(T, \mu_F) = A_0$ will be $^*t_M; {}^*f(t)$, where $\alpha_S(^*t_M) = \min_{f_l(t) \in D} \alpha_S^l(^*t_M^l)$.

Example

Table 1
Components with two groups of failure distributions

Distribution types	Parameter values
A. $f(t) = \frac{a_1 a_2}{a_2 - a_1}(e^{-a_1 t} - e^{-a_2 t}), \quad a_2 > a_1 > 0$	$a_1 \in \{0.5, 0.65, 0.8, 0.95, 1.1, 1.25, 1.4, 1.55\}$ $a_2/a_1 \in \{1.05, 1.3, 1.55, 1.8, 2.05\}$
B. $f(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} e^{-(t/\theta)^\beta}, \quad \theta > 0, \quad \beta > 0$	$\theta \in \{0.5, 1.0, 1.5, 2.0, 2.5\}$ $\beta \in \{0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0\}$

Table 2

Cost forms of maintenances and system unavailability

Cost forms	Parameter values
$C_M = \omega \mu_F e^{-\eta t_M}, \quad \omega > 0, \ \eta > 0$	$\omega = 1.0, \ \eta = 1.0$
$C_U = u e^{v(t_M - t_F)}, \quad u > 0, \ v > 0$	$u \in \{0.1, 1.0, 10.0\}, \ v \in \{0.25, 1.0, 5.0\}$

Table 3
Optimal solutions

$C_U = ue^{-v(t_M - t_F)}$		A_0				
		0.91	0.93	0.95	0.97	0.99
$u = 0.1$	A	$t_M^* = 2.529$	$t_M^* = 2.263$	$t_M^* = 1.963$	$t_M^* = 1.597$	$t_M^* = 1.052$
$v = 0.25$		$\alpha_S^* = 0.091$	$\alpha_S^* = 0.114$	$\alpha_S^* = 0.148$	$\alpha_S^* = 0.208$	$\alpha_S^* = 0.353$
	B	$t_M^* = 2.196$	$t_M^* = 2.066$	$t_M^* = 1.911$	$t_M^* = 1.707$	$t_M^* = 1.354$
		$\alpha_S^* = 0.122$	$\alpha_S^* = 0.135$	$\alpha_S^* = 0.154$	$\alpha_S^* = 0.185$	$\alpha_S^* = 0.260$
$u = 0.1$	A	$t_M^* = 2.529$	$t_M^* = 2.263$	$t_M^* = 1.963$	$t_M^* = 1.597$	$t_M^* = 1.052$
$v = 1.0$		$\alpha_S^* = 0.138$	$\alpha_S^* = 0.147$	$\alpha_S^* = 0.170$	$\alpha_S^* = 0.221$	$\alpha_S^* = 0.358$
	B	$t_M^* = 2.198$	$t_M^* = 2.067$	$t_M^* = 1.913$	$t_M^* = 1.708$	$t_M^* = 1.356$
		$\alpha_S^* = 0.134$	$\alpha_S^* = 0.144$	$\alpha_S^* = 0.160$	$\alpha_S^* = 0.190$	$\alpha_S^* = 0.261$
$u = 0.1$	A	$t_M^* = 0.839$	$t_M^* = 0.839$	$t_M^* = 0.839$	$t_M^* = 0.839$	$t_M^* = 0.839$
$v = 5.0$		$\alpha_S^* = 0.517$	$\alpha_S^* = 0.517$	$\alpha_S^* = 0.517$	$\alpha_S^* = 0.517$	$\alpha_S^* = 0.517$
	B	$t_M^* = 1.422$	$t_M^* = 1.422$	$t_M^* = 1.422$	$t_M^* = 1.422$	$t_M^* = 1.354$
		$\alpha_S^* = 0.286$	$\alpha_S^* = 0.286$	$\alpha_S^* = 0.286$	$\alpha_S^* = 0.286$	$\alpha_S^* = 0.289$

$u = 1.0$	A	$t_M^* = 2.529$	$t_M^* = 2.263$	$t_M^* = 1.963$	$t_M^* = 1.597$	$t_M^* = 1.052$
$v = 0.25$		$\alpha_S^* = 0.260$	$\alpha_S^* = 0.265$	$\alpha_S^* = 0.280$	$\alpha_S^* = 0.316$	$\alpha_S^* = 0.423$
	B	$t_M^* = 1.707$	$t_M^* = 1.707$	$t_M^* = 1.707$	$t_M^* = 1.707$	$t_M^* = 1.354$
		$\alpha_S^* = 0.268$	$\alpha_S^* = 0.268$	$\alpha_S^* = 0.268$	$\alpha_S^* = 0.268$	$\alpha_S^* = 0.294$
$u = 1.0$	A	$t_M^* = 1.425$	$t_M^* = 1.425$	$t_M^* = 1.425$	$t_M^* = 1.425$	$t_M^* = 1.052$
$v = 1.0$		$\alpha_S^* = 0.434$	$\alpha_S^* = 0.434$	$\alpha_S^* = 0.434$	$\alpha_S^* = 0.434$	$\alpha_S^* = 0.466$
	B	$t_M^* = 1.547$	$t_M^* = 1.547$	$t_M^* = 1.547$	$t_M^* = 1.547$	$t_M^* = 1.354$
		$\alpha_S^* = 0.293$	$\alpha_S^* = 0.293$	$\alpha_S^* = 0.293$	$\alpha_S^* = 0.293$	$\alpha_S^* = 0.304$
$u = 1.0$	A	$t_M^* = 0.452$	$t_M^* = 0.452$	$t_M^* = 0.452$	$t_M^* = 0.452$	$t_M^* = 0.452$
$v = 5.0$		$\alpha_S^* = 0.751$	$\alpha_S^* = 0.751$	$\alpha_S^* = 0.751$	$\alpha_S^* = 0.751$	$\alpha_S^* = 0.751$
	B	$t_M^* = 1.045$	$t_M^* = 1.045$	$t_M^* = 1.045$	$t_M^* = 1.045$	$t_M^* = 1.045$
		$\alpha_S^* = 0.412$	$\alpha_S^* = 0.412$	$\alpha_S^* = 0.412$	$\alpha_S^* = 0.412$	$\alpha_S^* = 0.412$

$u = 10.0$	A	$t_M^* = 0.331$	$t_M^* = 0.331$	$t_M^* = 0.331$	$t_M^* = 0.331$	$t_M^* = 0.331$
$v = 0.25$		$\alpha_S^* = 0.949$	$\alpha_S^* = 0.949$	$\alpha_S^* = 0.949$	$\alpha_S^* = 0.949$	$\alpha_S^* = 0.949$
	B	$t_M^* = 0.942$	$t_M^* = 0.942$	$t_M^* = 0.942$	$t_M^* = 0.942$	$t_M^* = 0.942$
		$\alpha_S^* = 0.481$	$\alpha_S^* = 0.481$	$\alpha_S^* = 0.481$	$\alpha_S^* = 0.481$	$\alpha_S^* = 0.481$
$u = 10.0$	A	$t_M^* = 0.201$	$t_M^* = 0.201$	$t_M^* = 0.201$	$t_M^* = 0.201$	$t_M^* = 0.201$
$v = 1.0$		$\alpha_S^* = 0.967$	$\alpha_S^* = 0.967$	$\alpha_S^* = 0.967$	$\alpha_S^* = 0.967$	$\alpha_S^* = 0.967$
	B	$t_M^* = 0.898$	$t_M^* = 0.898$	$t_M^* = 0.898$	$t_M^* = 0.898$	$t_M^* = 0.898$
		$\alpha_S^* = 0.493$	$\alpha_S^* = 0.493$	$\alpha_S^* = 0.493$	$\alpha_S^* = 0.493$	$\alpha_S^* = 0.493$
$u = 10.0$	A	$t_M^* = 0.066$	$t_M^* = 0.066$	$t_M^* = 0.066$	$t_M^* = 0.066$	$t_M^* = 0.066$
$v = 5.0$		$\alpha_S^* = 0.989$	$\alpha_S^* = 0.989$	$\alpha_S^* = 0.989$	$\alpha_S^* = 0.989$	$\alpha_S^* = 0.989$
	B	$t_M^* = 0.703$	$t_M^* = 0.703$	$t_M^* = 0.703$	$t_M^* = 0.703$	$t_M^* = 0.703$
		$\alpha_S^* = 0.552$	$\alpha_S^* = 0.318$	$\alpha_S^* = 0.318$	$\alpha_S^* = 0.318$	$\alpha_S^* = 0.318$

Conclusions

- This property is true for any failure distributions of the component, the constraint of system availability can be always relaxed.
- This decision variable is different from the others in that it is a function.
- This approach can also be applied to those problems that treat discrete cost values.

The End

Thanks!