A simple and effective R chart to monitor the process variance

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Keywords: Average run length; Control charts

Introduction (1/3)

- 管制圖為用來監視操作中製程是否處於被預期變異之內，其製程變異可經由樣本變異數或樣本全距觀察而得。因此對於監視製程變異程度可加以利用R,S和S^2 charts等管制圖。Montgomery(1991)建議對於連續性製程，應同步針對其製程平均及變異數來加以保持監視。Ryan(1989)更主張製程產業中，其R chart更應優於 $\overline{X}$ chart來先探討。

Introduction (2/3)

- 文獻回顧：
  - Chang and Gan(1995)發展一樣本變異數經由對數轉換過程log(S^2)之CUSUM管制圖，且假設log(S^2)之分配近似常態。對於監視製程變異小偏移之執行能力方面，具有one-sided或two-sided設計程序之CUSUM管制圖可得到幾近最佳，且指出若當製程起始態為管制內，若加入FIR特色之CUMSU管制圖，其初始值(head start value)設為零，亦較不易受影響。

Introduction (3/3)

- Chang and Gan(2004)提出對於監視製程變異成份之Shewhart管制圖，並推導一方程式為其變異成分樣本統計量之機率分配。認為當假設目標標準差為1，若偏移至三個製程標準差時，所計算出ARL將犯下型一誤差機率為0.002。

Proposed R chart (1/8)

- 原理：假設製程目標平均與標準差分別為μ_0與σ_0，但在任一時間其真實標準差為σ_1，管制檢定如下：

  Null hypothesis(H_0)： $\mu=\mu_0$ and $\sigma=\sigma_0$
  Alternative hypothesis(H_1)： $\mu=\mu_0$ and $\sigma\neq\sigma_0$

  由於Shewhart R chart對於製程標準差小偏移之偵測方面有所受限，故一近似Shewhart R chart，具有警告界限及管制界限之proposed R chart被提出。

Proposed R chart (2/8)

- 模擬執行：Box and Muller(1958)曾建議，若需產生一具有平均數$\mu_0 = 0$及標準差$\sigma_0 = 1$之X觀察值數列，則可使用下式：

$$X = \sqrt{-2\log r} \times \mathrm{Cos}(2\pi r_2)$$

$$Y = \mathrm{am} + \mathrm{asd}^{*} X$$

r_1與r_2為介於0～1之隨機變數

Am = actual mean of the process

Asd = actual standard deviation of the process

且Prajapati(2006)利用卡方檢定證實，在99%信心水準下，其上式所產生之數列符合特定平均術與標準差之常態分配。

Proposed R chart (3/8)

- 設計程序：
  1. 基於隨機原理抽取樣本
  2. 繪製樣本全距點於R chart上，若有任一點超出上管制界限(UCL_R)，則對立假設將立即接受。
  3. 若任一點座落於上警告界限(UWL_R)與上管制界限之間，則計算其統計量U。
  4. 計算得統計量U，若大於在某顯著水準下，一具有自由度n x H之卡方分配特定值U*，則其製程為管制外且應立即修正必要行動。

H : history, ie. Number of preceding observations considered for calculating statostic, U

Proposed R chart (4/8)

- 理論：當一製程標準差從σ_0增加為$\sigma_1(=r\sigma_0)$，定義P(r)為管制外訊號顯示之機率函數，其表示為：

$$P(r) = f(n, K, L, H \text{and} U^*) = \Pr{}_1(r) + \Pr{}_2(r) \times \Pr{}_3(r)$$

$$= \{1 - \alpha(n, L)\} + \{\alpha(n, L) - \alpha(n, K)]\} \times \left[1 - \boxed{\beta\left(nH, \frac{U}{r^2}\right)}\right]$$

$$U = \sum_{i=1}^{H}\sum_{j=1}^{n}\left(\frac{Xij - \mu_0}{\sigma}\right)^2$$

$$= \sum_{i=1}^{H}\sum_{j=1}^{n}\left(\frac{Xij - \mu_0}{\sigma_1/r}\right)^2$$

$$= r^2\sum_{i=1}^{H}\sum_{j=1}^{n}\left(\frac{Xij - \mu_0}{\sigma_1}\right)^2$$

$$\Pr{}_3(r) = P\left(r^2\sum_{i=1}^{H}\sum_{j=1}^{n}\left(\frac{Xij - \mu_0}{\sigma_1}\right)^2 < U^*\right)$$

$$= P\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{Xij - \mu_0}{\sigma_1}\right)^2 < \frac{U^*}{r^2}\right)$$

Proposed R chart (5/8)

Example:

給定條件

Sample size(n) = 4, History(H) = 4, Degrees of freedom(υ) = n x H = 16

Confidence level = 95% , 查表得U* = 26.3, d_2 = 2.059, D_4 = 2.28

in-control ARL = 500下 ，L = 3.6, K = 2.4

計算得

Average sample range($\overline{R}$) = $d_2 \times \sigma_0$ = 2.059 x 1 = 2.05

Estimate of standard deviation(σ_R) = D4($\overline{R}$ - 1)/3 = 0.8

Upper control limit on R chart(UCL_R) = $\overline{R} + L \times \sigma_R$ = 5.22

Upper warning limit on R chart(UWL_R) = $\overline{R} + K \times \sigma_R$ = 4.17

Proposed R chart (6/8)

S. no.	X1	X2	X3	X4	Range
1	−0.908	0.371	0.333	1.212	2.12
2	−2.083	−1.165	1.019	−0.638	3.102
3	0.613	−1.166	−1.551	1.851	3.402
4	−1.014	2.601	0.042	1.242	3.615
5	−0.532	1.363	0.77	−0.128	1.895
6	2.64	−1.176	2.774	0.54	3.95
					4.298
					1.367
					1.812
10	−0.570	−1.600	−1.681	−0.335	1.346
11	0.998	−0.453	0.261	0.174	1.451
12	0.105	−1.433	1.47	0.082	2.903
13	−0.088	0.506	1.894	0.934	1.982
					4.547
					3.6
					4.366
17	−2.592	2.471	−1.013	−0.029	5.063
18	0.669	−0.952	1.861	1.288	2.813

U = {[(0.613 - 0.0) / 1.0]^2 + [(-1.166 - 0.0) / 1.0]^2 + --- + [(0.54 - 0.0) / 1.0]^2}
= 35.994 > 26.3 => out of control

U = {[(-0.570 - 0.0) / 1.0]^2 + [(-1.600 - 0.0) / 1.0]^2 + --- + [(0.934 - 0.0) / 1.0]^2}
= 16.078 < 26.3 => in control

Proposed R chart (7/8)

Sample size (n)	L	K	H	U*	Factor d_2	Factor D_3	Factor D_4	Upper control limits (UCL_R)	Upper warning Limits (UWL_R)
2	3.9	2.8	4	15.51	1.128	0.00	3.27	(4.46)	(3.52)
3	3.7	2.5	4	21.03	1.693	0.00	2.57	4.794	3.73
4	3.6	2.4	4	26.3	2.059	0.00	2.282	(5.05)	(4.17)
5	3.6	2.5	4	31.41	2.326	0.00	2.11	(6.03)	(4.90)
10	3.5	2.5	4	55.76	3.078	0.00	1.78	(5.88)	(5.08)

Table I.
Parameters of proposed R chart for in-control ARL of 500

Proposed R chart (8/8)

Shift ratio $r = \sigma_1/\sigma_0$	Proposed R chart ARLs	S.D. of run length
1.00	500.00	10.52
1.07	168.00	6.38
1.13	78.00	4.21
1.17	47.50	2.32
1.20	41.20	1.85
1.27	19.60	1.64
1.33	12.80	1.46
1.40	8.90	1.12
1.47	6.80	1.11
1.50	5.90	1.02
1.53	5.30	0.86
1.60	4.30	0.72
1.67	3.70	0.68
1.73	3.20	0.26
1.80	2.80	0.102
1.83	2.70	0.091
1.87	2.60	0.083
1.93	2.40	0.054
2.00	2.20	0.041
2.17	1.90	0.022
2.33	1.70	0.021
2.50	1.56	0.018
3.00	1.31	0.016

Table II.
ARLs of proposed R chart for sample size of four

Performance comparisons (1/4)

Sample size (n)	Shift ratio (r) (σ_1/σ_0)	Parameters (CUSUM chart) k	h	ARLc (Chang's chart)	ARLp (Proposed R chart)	ARL Ratio = ARLp/ ARLc
2	1.00	0.541	1.106	500	500	1.00
2	1.20	0.900	1.597	85.6	64.1	0.75
2	1.60	0.972	1.471	15.7	10.5	0.67
2	2.20	1.085	1.294	5.6	4.2	0.75
2	3.00	1.204	1.130	3.1	2.4	0.77
3	1.00	0.541	1.106	500	500	1.00
3	1.20	−0.213	7.103	45.5	44.0	0.97
3	1.60	0.679	1.409	8.7	5.9	0.68
3	2.20	0.879	1.052	3.2	2.2	0.69
3	3.00	1.039	0.834	1.9	1.5	0.79
5	1.00	0.541	1.106	500	500	1.00
5	1.20	0.011	3.961	24.9	39	1.56
5	1.60	0.541	1.106	4.8	4.3	0.89
5	2.20	0.770	0.722	2.0	1.8	0.90
5	3.00	0.952	0.504	1.3	1.2	0.92
10	1.00	0.541	1.106	500	500	1.00
10	1.20	0.126	1.957	12.7	22.5	1.77
10	1.60	0.482	0.656	2.5	2.6	1.04
10	2.20	0.710	0.362	1.2	1.2	1.00
10	3.00	0.800	0.266	1.0	1.0	1.00

Table III. ARLs comparison of proposed R chart with Chang's CUSUM charts (without FIR) for various sample sizes (n)

Performance comparisons (2/4)

Sample size (n)	Shift Ratio ($r = \sigma_1/\sigma_0$)	Parameters (CUSUM chart) k	h	Chang's chart	Proposed R chart	ARLs Ratio
2	1.00	0.541	1.106	500	500	1.00
2	1.20	0.900	1.605	81.7	64.1	0.78
2	1.60	0.972	1.477	13.9	10.5	0.76
2	2.20	1.085	1.299	4.8	4.2	0.88
2	3.00	1.204	1.133	2.7	2.4	0.89
3	1.00	0.541	1.106	500	500	1.00
3	1.20	−0.213	7.240	34.4	44.0	1.28
3	1.60	0.679	1.417	7.0	5.9	0.84
3	2.20	0.879	1.056	2.7	2.2	0.81
3	3.00	1.039	0.836	1.7	1.5	0.88
5	1.00	0.541	1.106	500	500	1.00
5	1.20	0.011	4.023	17.6	39	2.22
5	1.60	0.541	1.112	3.6	4.3	1.19
5	2.20	0.770	0.725	1.6	1.8	1.13
5	3.00	0.952	0.504	1.3	1.2	0.92
10	1.00	0.541	1.106	500	500	1.00
10	1.20	0.126	1.979	8.6	22.5	2.62
10	1.60	0.482	0.482	1.9	2.6	1.37
10	2.20	0.710	0.710	1.1	1.2	1.09
10	3.00	0.800	0.266	1.0	1.0	1.00

Table IV.
ARLs comparison of proposed R chart with ARLs of Chang's CUSUM charts (with FIR) for various sample sizes (n)

Performance comparisons (3/4)

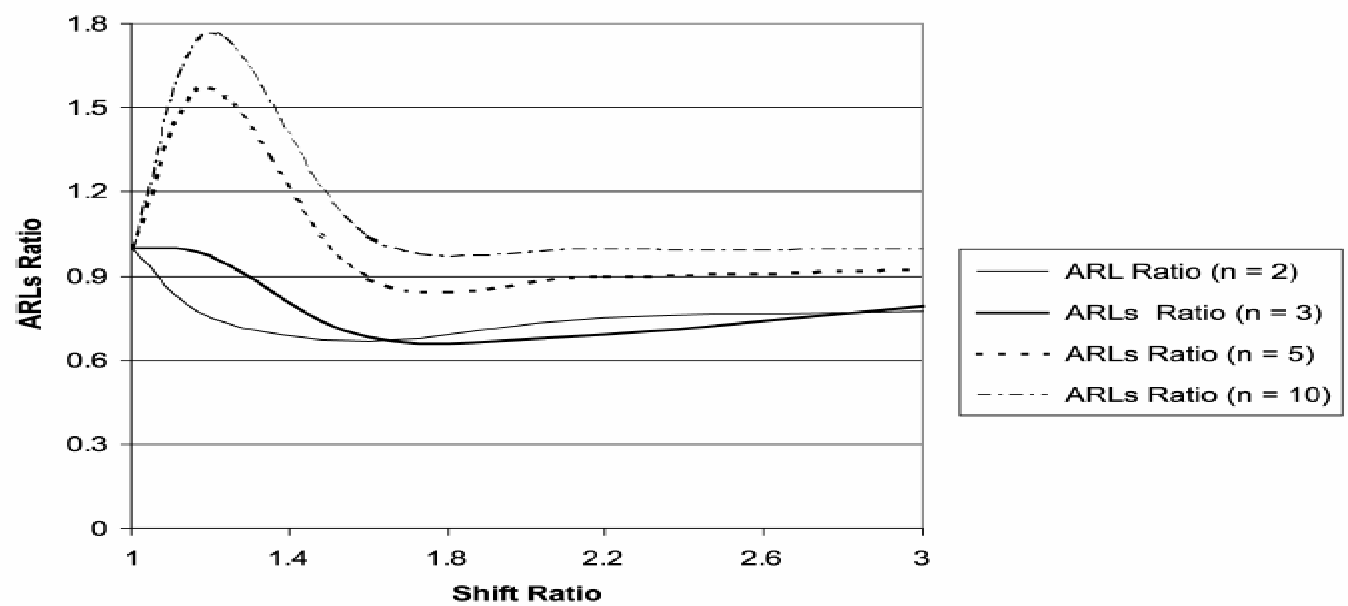

Figure 1.
Plot for ARLs ratio between proposed R chart and Chang's CUSUM charts (without FIR) for various sample sizes (n)

- Proposed R chart與Chang’s CUSUM charts(with/without FIR)管制內之誤警率(false alarm rate)相等。
- 由大多數case觀察得，Proposed R chart對於Chang’sCUSUM charts(with/without FIR)具有較小ARL。
- Chang’s CUSUM chart加入FIR特色其執行能力獲得較小之改善。
- 由表三與表四之40個case顯示，Proposed R chart與Chang’s CUSUM charts (with/without FIR)之ARLs ratio超過1共10個case。

Performance comparisons (4/4)

S. No.	Shift ratio ($r = \sigma_1/\sigma_0$)	Chang's Chart (1) UCL = 7.966 (appx. limit)	Chang's Chart (2) UCL = 5.322 (appx. limit)	Chang's Chart (3) UCL = 4.301 (exact limit)	K= 2.5 L = 3.7 Proposed R chart	ARLs ratio (1)	ARLs ratio (2)	ARLs ratio (3)
1.	1.00	500.00	500.00	500.00	500.00	1.00	1.00	1.00
2.	1.05	321.05	292.31	272.6	245.0	0.76	0.84	0.90
3.	1.10	214.88	180.42	158.58	127.1	0.59	0.70	0.80
4.	1.15	149.26	116.86	97.71	73.5	0.49	0.63	0.75
5.	1.20	107.17	79.01	63.35	44.0	0.41	0.56	0.69
6.	1.25	79.26	55.50	42.97	30.1	0.38	0.54	0.70
7.	1.30	60.19	40.33	30.34	21.2	0.35	0.53	0.70
8.	1.35	46.80	30.21	22.20	16.1	0.34	0.53	0.73
9.	1.40	37.17	23.24	16.76	13.0	0.35	0.56	0.78
10.	1.45	30.09	18.32	13.02	9.9	0.33	0.54	0.76
11.	1.50	24.78	14.76	10.37	8.3	0.33	0.56	0.80
12.	1.60	17.57	10.12	7.01	5.9	0.34	0.58	0.84
13.	1.70	13.11	7.39	5.10	4.7	0.36	0.64	0.92
14.	1.80	10.19	5.68	3.92	3.9	0.38	0.69	0.99
15.	1.90	8.19	4.55	3.16	3.2	0.39	0.70	1.01
16.	2.00	6.78	3.76	2.64	2.8	0.41	0.74	1.06
17.	2.50	3.53	2.05	1.55	1.9	0.54	0.93	1.23
18.	3.00	2.43	1.52	1.23	1.5	0.62	0.99	1.22

- Proposed R chart與Chang's upper-sided Shewhart variance chart管制內之誤警率(false alarm rate)相等。
- 由多數製程偏移比觀察得，Proposed R chart對於Chang's upper-sided Shewhart variance chart具有較小ARL。

Concluding remarks

- 當製程變異為單一不變偏移時，且對於CUSUM chart操作或設計上所產生眾多困難之處，可經由一簡單及準確之proposed R chart計畫來加以簡化，因其管制圖不需思考眾多參數性質，故適用在單樣本數與單管制界限中。

相關文獻：

D.R. Prajapati and P.B. Mahapatra(2008), "A simple and effective $\overline{X}$ chart to process monitoring", International Journal of Quality & Reliability Management, Vol. 25 No. 5, pp. 508-531

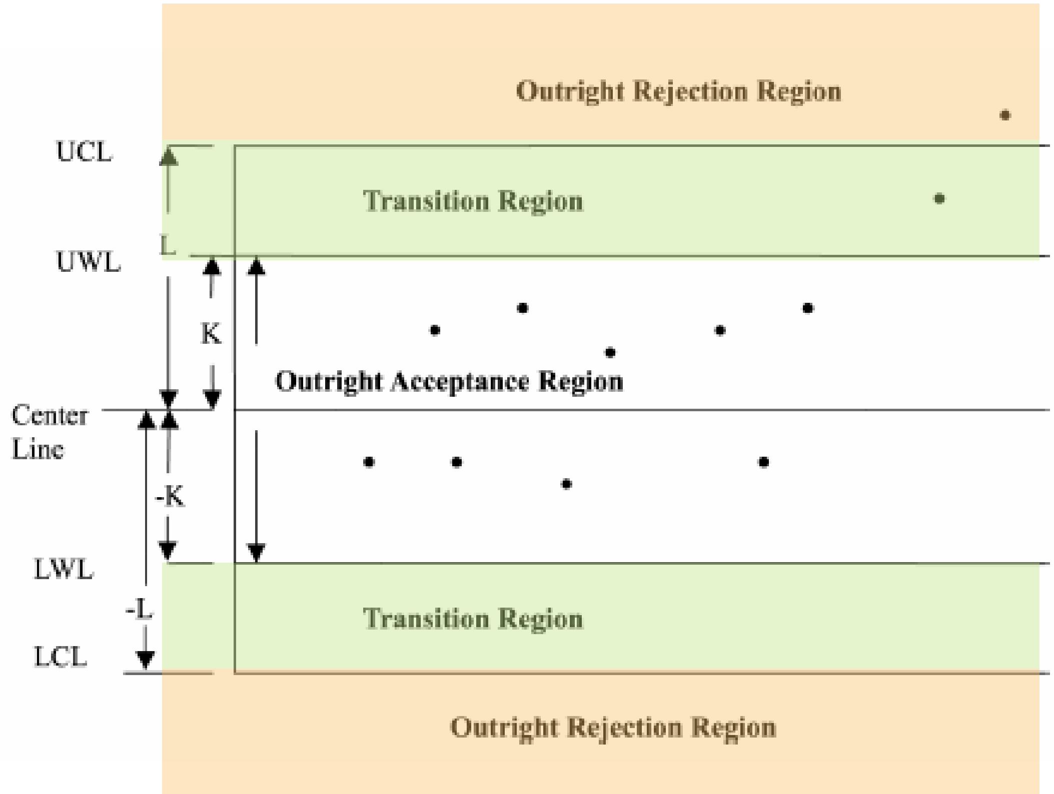

Figure 1.
Pattern of points falling in different regions of new $\bar{X}$ chart

當一製程平均偏移量為 δ ，定義P(δ)為管制外訊號顯示之機率函數，其表示為：

$$P(\delta) = f\left(n, K, L, H, U\right) = \Pr{}_1(\delta) + \Pr{}_2(\delta) * \Pr{}_3(\delta)$$

$$= \left[1 - \{\alpha(L-\delta) + \alpha(-L-\delta)\}\right]$$

$$+ \left[\alpha\{(L-\delta) - (K-\delta)\} + \alpha\{(-L-\delta) - (K-\delta)\}\right]$$

$$\times P\left(U > U^* - v \times \delta^2/n\right)$$

Strategy based on sum of Chi-squares; CSQ

$$= \left[1 - \{\alpha(L-\delta) + \alpha(-L-\delta)\}\right]$$

$$+ \left[\alpha\{(L-\delta) - (K-\delta)\} + \alpha\{(-L-\delta) - (K-\delta)\}\right]$$

$$\times \left[1 - \{\alpha((L-\delta)/\bar{\sigma})\} - \{\alpha((-L-\delta)/\bar{\sigma})\}\right]$$

Strategy based on average of sample means; ASM

$$\text{average standard deviation of "H" samples} = \bar{\sigma} = \frac{\sigma'}{\sqrt{H^* n}}$$

	Strategy CSQ		Strategy ASM
	ARLs (Simulated)	ARLs (theoretical)	ARLs (theoretical)
0.0	373	371.0968	377.48
0.2	308	311.4167	245.83
0.4	201	202.0513	116.38
0.5	152	154.9996	76.68
0.6	115	117.0993	50.29
0.8	64	65.47666	22.24
1.0	35	36.31142	10.63
1.2	20.3	20.17047	5.66
1.4	10.9	11.31934	3.38
1.5	8.7	8.551285	2.73
1.6	6.3	6.522352	2.27
1.8	3.9	3.973474	1.69
2.0	2.5	2.666285	1.38

Table III.
Comparison of simulated ARLs with theoretical ARLs for both strategies

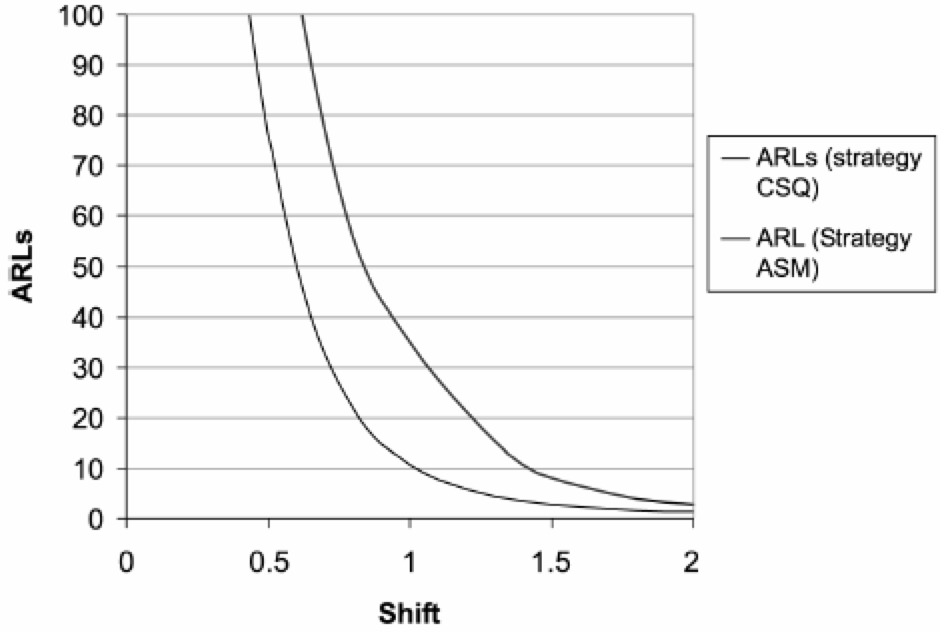

Figure 2.
Plot between theoretical performance of new chart, using strategy CSQ and strategy ASM

Sets	Charts	(n_1, n_2)	(h_1, h_2)	(K_1, K_2)	Shift 0.0	0.50	1.0	1.5	2.0	3.0
1st	CUSUM chart	(4)	–	–	370	25.1	9.4	5.6	4.0	2.3
	VP $\bar{X}$ chart	(1, 14)	(3.96, 0.17)	(4, 1.96)	370	25.0	8.8	5.2	3.5	2.3
2nd	CUSUM chart	(4)	–	–	370	68.0	12.8	4.8	2.7	1.3
	V P $\bar{X}$ chart	(1, 4)	(1.43, 0.13)	(4, 2.12)	370	61.7	8.6	2.9	1.7	1.0
–	New $\bar{X}$ chart (Strategy ASM)	$n = 1$ and $H = 4$	(1.00, 1.00)	$L = 3.05$ $K = 1.0$	379	77.0	10.7	2.7	1.3	1.00
–	New $\bar{X}$ chart (Strategy CSQ)	$n = 4$ and $H = 4$	(1.00, 1,00)	$L = 3.2$ $K = 2.2$	373	152	35	8.5	2.6	1.25

Table VI.
Comparison of performance of new $\bar{X}$ chart with CUSUM and VP $\bar{X}$ chart

Shift (δ)	Lucas's (Shewhart-CUSUM) scheme	New $\bar{X}$ chart (Strategy ASM)	New $\bar{X}$ chart (Strategy CSQ)
0.0	448	379	373
0.5	34.8	77	152
0.75	16.1	28	76
1.0	9.84	10.7	35
1.5	5.29	2.7	8.5
2.0	3.46	1.3	2.6
2.5	2.44	1.1	1.6
3.0	1.81	1.0	1.25

Table VII.
ARLs comparison between new chart and Lucas's schemes