Economic design of two-stage $\bar{X}$ charts: The Markov chain approach

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Keywords: Economic design; Two-stage control; Performance variable; Surrogate variable

Introduction

- 本篇paper 將呈現以績效變數(performance variable)和替代變數(surrogate variable)去建立兩階段(two-stage)經濟性設計管制圖之模式。

  績效變數(X)：測量所感興趣之品質特性成本

  替代變數(Y)：與績效變數存在相關性，但其花費成本為較少

- 製程監控從 $\bar{Y}$ 管制圖開始，直到 $\bar{Y}$ 樣本平均落於管制限界外時，則轉移至 $\bar{X}$ 管制圖。$\bar{X}$ 管制圖由中心(central)、警告(warning)和行動(action) 區域所組成。當　樣本平均落於中心線附近時，則製程監控返回 $\bar{Y}$ 管制圖，否則製程仍舊在 $\bar{X}$ 管制圖監控下，直到 $\bar{X}$ 樣本平均落於管制限界外，此時可歸屬原因可著手尋找

Nomenclature

- n_x sample size for the $\bar{X}$ chart
- n_y sample size for the $\bar{Y}$ chart
- h_x sampling interval length for the $\bar{X}$ chart
- h_y sampling interval length for the $\bar{Y}$ chart
- k_x control limit factor for the $\bar{X}$ chart
- k_y control limit factor for the $\bar{Y}$ chart
- w_x warning limit factor for the $\bar{X}$ chart
- σ_X standard deviation of the performance variable
- σ_Y standard deviation of the surrogate variable
- μ_X mean of the performance variable, before the process shift
- $\mu_X \pm c\sigma_X$ mean of the performance variable, after the process shift
- μ_Y mean of the surrogate variable, before the process shift
- $\mu_Y \pm \beta_1 c\sigma_Y$ mean of the surrogate variable, after the process shift
- β_1 coefficient proportional to the covariance between X and Y
- λ^{-1} interval length the process stays in control
- a_1 the cost of finding and eliminating an assignable cause that occurs once during a cycle
- a_2 the cost of investigating a false alarm
- a_3 the fixed cost of sampling and testing for X variables
- a_4 the variable cost of sampling and testing for X variables
- a'_3 the fixed cost of sampling and testing for Y variables
- a'_4 the variable cost of sampling and testing for Y variables
- b_1 the expected time required to find and eliminate the assignable cause
- b_2 the expected time wasted with false alarms
- b_3 the time required to take and interpret an X sample
- b'_3 the time required to take and interpret an Y sample
- i_1 income per hour when the process is in control
- i_2 income per hour when the process is out of control
- $E(T)$ expected cycle time
- $E(I)$ expected net income per cycle
- $E(FA)$ average number of false alarms per cycle
- $E(A)$ expected net income per unit time
- AT average time from the start of production until the first signal after the process shift
- m_x average number of X samples taken from the process before the process shift
- m_y average number of Y samples taken from the process before the process shift
- m'_x average number of X samples taken from the process after the process shift
- m'_y average number of Y samples taken from the process after the process shift

Description of the two-stage control chart (1/2)

$$X_i = \zeta_i + \delta_i \qquad \delta_i \sim N(0, \sigma_\delta^2)$$

$$Y_i = \beta_0 + \beta_1 \zeta_i + \varepsilon_i \qquad \varepsilon_i \sim N(0, \sigma_\varepsilon^2)$$

$\zeta i \sim N(\zeta, \sigma_\zeta^2)$ 故 $\delta_i \perp \varepsilon_i$ 且 $\zeta_i \perp \delta_i$ 和 ε_i (Fuller,1987)

關於 (X, Y) 二元常態向量

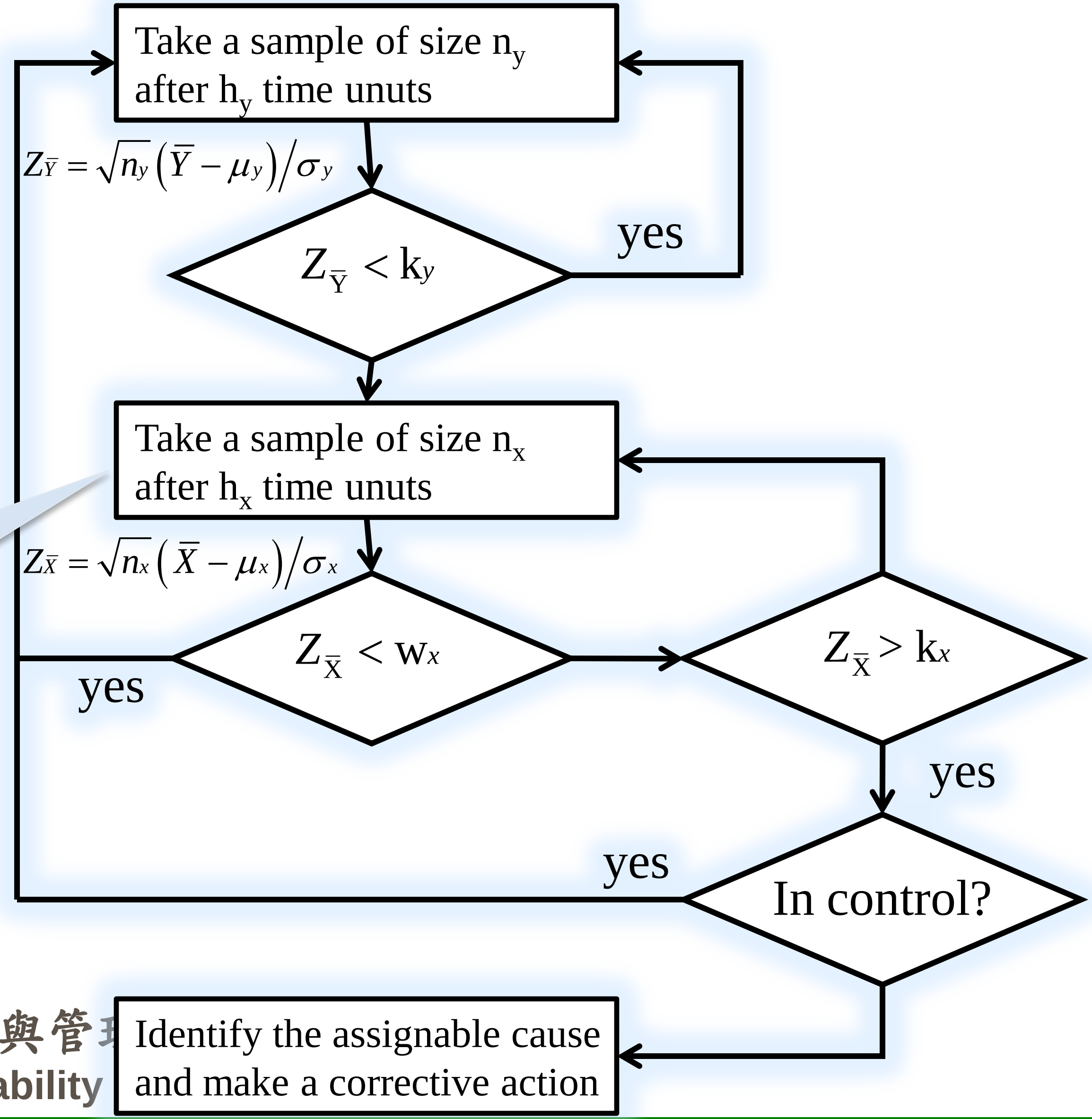
Mean: $(\mu_x, \mu_y) = (\zeta, \beta_0 + \beta_1 \zeta)$

Covariance matrix: $\begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} \sigma_\delta^2 + \sigma_\zeta^2 & \beta_1 \sigma_\zeta^2 \\ \beta_1 \sigma_\zeta^2 & \beta_1^2 \sigma_\zeta^2 + \sigma_\varepsilon^2 \end{pmatrix}$

$\beta 1$ 估計

$$\hat{\beta} 1 = \begin{cases} \left\{ \left(S_X^2 - \sigma_\delta^2 \right)^{-1} S_{XY} \right\} & \text{if } S_Y^2 \left(S_X^2 - \sigma_\delta^2 \right) - S_{XY}^2 > 0 \\ S_Y^2 \big/ S_X^2 & \text{otherwise} \end{cases}$$

Description of the two-stage control chart (2/2)



$\bar{X}$ chart for a previously specified amount of time: $T_x = mh_x$,

m is an integer fixed in advance

(Lee and Kwon, 1999)

Properties of the two-stage control chart (1/5)

- 製造程序因停止去找尋和修護時間可忽略
- 可歸屬原因在兩階段管制圖過渡期間，僅只發生在其任一階段
- 因指數分配具有無記憶性(memoryless)容許使用馬可夫鏈得兩階段管制圖之特性

(Costa and Rahim, 2004)

Properties of the two-stage control chart (2/5)

Table 1
The Markov chain one-step state transitions

*i*th sample				(*i* + 1)th sample	
Chart in use	Process mean status[a] (on or off target)	Sample point position (region)	State of the Markov chain	Process mean status	State of the Markov chain
$\bar{Y}$	On	Central	1	On	1
$\bar{Y}$	On	Action	1	On	2
$\bar{Y}$	On	Central	1	Off	3
$\bar{Y}$	On	Action	1	Off	4
$\bar{X}$	On	Central	2	On	1
$\bar{X}$	On	Action	2	On	1
$\bar{X}$	On	Warning	2	On	2
$\bar{X}$	On	Central	2	Off	3
$\bar{X}$	On	Action	2	Off	3
$\bar{X}$	On	Warning	2	Off	4
$\bar{Y}$	Off	Central	3	Off	3
$\bar{Y}$	Off	Action	3	Off	4
$\bar{X}$					3
$\bar{X}$					4
$\bar{X}$					

[a]On target mea

[b]True alarm oc

state 1 : The process is in control and under $\bar{Y}$ chart surveillance

state 2 : The process is in control and under $\bar{X}$ chart surveillance

state 3 : The process is out of control and under $\bar{Y}$ chart surveillance

state 4 : The process is out of control and under $\bar{X}$ chart surveillance

state 5 : The absorbing state

Properties of the two-stage control chart (3/5)

The transition probability matrix $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & 0 \\ p_{21} & p_{22} & p_{23} & p_{24} & 0 \\ 0 & 0 & p_{33} & p_{34} & 0 \\ 0 & 0 & p_{43} & p_{44} & p_{45} \\ 0 & 0 & 0 & 0 & p_{55} \end{bmatrix}$

$$p_{11} = \Pr[|Z| < k_y] e^{-\lambda h_y},$$

$$p_{12} = \Pr[|Z| > k_y] e^{-\lambda h_x},$$

$$p_{13} = \Pr[|Z| < k_y](1 - e^{-\lambda h_y}),$$

$$p_{14} = \Pr[|Z| > k_y](1 - e^{-\lambda h_x}),$$

$$p_{21} = \{1 - \Pr[w_x < |Z| < k_x]\} e^{-\lambda h_y},$$

$$p_{22} = \Pr[w_x < |Z| < k_x] e^{-\lambda h_x},$$

$$p_{23} = \{1 - \Pr[w_x < |Z| < k_x]\}(1 - e^{-\lambda h_y}),$$

$$p_{24} = \Pr[w_x < |Z| < k_x](1 - e^{-\lambda h_x}),$$

$$p_{33} = \Pr[|U_y| < k_y],$$

$$p_{34} = 1 - p_{33},$$

$$p_{43} = \Pr[|U_x| < w_x],$$

$$p_{44} = \Pr[w_x < |U_x| < k_x],$$

$$p_{45} = 1 - p_{43} - p_{44},$$

$$p_{55} = 1,$$

where $Z \sim N(0, 1)$, $U_x \sim N(c\sqrt{n_x}, 1)$, and $U_y \sim N(\beta_1 c\sqrt{n_y}, 1)$.

Properties of the two-stage control chart (4/5)

$$(I-Q)^{\;1} = \begin{bmatrix} \dfrac{1-p_{22}}{A} & \dfrac{p_{12}}{A} & \dfrac{C(1-p_{44})+Dp_{43}}{AB} & \dfrac{Cp_{34}+D(1-p_{33})}{AB} \\ \dfrac{p_{21}}{A} & \dfrac{1-p_{11}}{A} & \dfrac{E(1-p_{44})+Fp_{43}}{AB} & \dfrac{Ep_{34}+F(1-p_{33})}{AB} \\ 0 & 0 & \dfrac{1-p_{44}}{B} & \dfrac{p_{34}}{A} \\ 0 & 0 & \dfrac{p_{43}}{B} & \dfrac{1-p_{33}}{B} \end{bmatrix},$$

where

$A - (1-p_{11})(1-p_{22}) - p_{12}p_{21},$

$B - (1-p_{44})(1-p_{33}) - p_{34}p_{43},$

$C - p_{23}p_{12} + p_{13}(1-p_{22}),$

$D - p_{24}p_{12} + p_{14}(1-p_{22}),$

$E - p_{13}p_{21} + p_{23}(1-p_{11})$

and

$F - p_{14}p_{21} + p_{24}(1-p_{11}).$

I: the identity matrix of order 4

Q: the transition probability matrix of order 4, where the elements associated with the absorbing state have been deleted

Properties of the two-stage control chart (5/5)

$$m_y = b'(I-Q)^{-1}c_1$$

$$m_x = b'(I-Q)^{-1}c_2$$

$$m_y = b'(I-Q)^{-1}c_3$$

$$m_x = b'(I-Q)^{-1}c_4$$

b': the vector of starting probabilities, constructed under the assumption that the process surveillance starts with the $\bar{Y}$ chart $= (e^{-\lambda h y};\ 0;\ 1\text{-}e^{-\lambda h y};\ 0)$

E(FA): the average number of false alarms per cycle $= m_x \Pr\left[|Z| > k_x\right]$

AT: the average time from the start of production until the first signal after the process shift

$$= b'(I-Q)^{-1}h',\ h' = (h_y;\ h_x;\ h_y;\ h_x)$$

The cost model (1/3)

The expected cycle time: $E(T)$

= the in-control period: $1/\lambda$

+ the out-of-control period: $AT - 1/\lambda + b_3 n_x + b_3^{'} n_y$

$\left(\begin{array}{l}\text{the expected time between the occurrence of an assignable cause} \\ \text{and the signal \& the time required to take and interpret the results}\end{array}\right)$

+ the period during which the process is stopped due to false alarms: $b_2 E(FA)$

+ the period during which the process is stopped for finding and eliminating an assignable: b_1

The cost model (2/3)

The expected net income per cycle: $E(I)$

= the income per hour while the process is in control times

the expected length of the in-control period: i_1 / λ

+ the income per hour while the process is out of control times

the expected length of the out-of-control period: $i_2 * \left(AT - 1/\lambda + b_3 n_x + b_3' n_y\right)$

– the cost of finding and eliminating an assignable cause: a_1

– the cost of investigating a false times the expected number of false alarms: $a_2 E(AF)$

– the expected sampling cost per cycle: $\left[\left(a_3 + a_4 n_x\right) * \left(m_x + m_x'\right) + \left(a_3' + a_4' n_y\right) * \left(m_y + m_y'\right)\right]$

The cost model (3/3)

The expected net income per unit time: $\mathrm{E}(A) = \mathrm{E}(I) / \mathrm{E}(T)$

Subject to:

$$b_3 n_x \le \eta h_x,$$

$$b_3' n_y \le \eta h_y,$$

$$0.01 \le h_x,\ h_y \le 20,\ 0.01 \le k_x,\ k_y \le 4.$$

Since the process is a renewal reward process *(Ross, 1992)*. The restrictions are imposed on the design parameters to accommodate limitations of practical order.

Comparison of one-stage and two-stage designs (1/2)

Table 2
Cost and process parameters for test examples

Example	λ	c	i_1	i_2	a_1	a_2	a_3	a_4	b_1	b_2	b_3
2	0.01	1	150	50	350	500	5.0	1.0	3.05	4.05	0.05
4	0.01	2	150	50	135	500	0.5	0.1	4.00	41.00	0.05
6	0.05	1	50	−50	350	50	0.5	1.0	4.00	41.00	0.05
8	0.05	2	50	−50	135	50	5.0	0.1	3.05	4.05	0.05
10	0.01	1	150	50	260	50	5.0	0.1	4.00	5.00	0.50
12	0.01	2	150	50	45	50	0.5	1.0	3.05	40.05	0.50
14	0.05	1	50	−50	260	500	0.5	0.1	3.05	40.05	0.50
16	0.05	2	50	−50	45	500	5.0	1.0	4.00	5.00	0.50
18	0.01	1	50	−50	45	500	5.0	1.0	20.05	40.05	0.05
20	0.01	2	50	−50	260	500	0.5	0.1	21.00	5.00	0.05
22	0.05	1	150	50	45	50	0.5	1.0	21.00	5.00	0.05
24	0.05	2	150	50	260	50	5.0	0.1	20.05	40.05	0.05
26	0.01	1	50	−50	135	50	5.0	0.1	21.00	41.00	0.50
28	0.01	2	50	−50	350	50	0.5	1.0	20.05	4.05	0.50
30	0.05	1	150	50	135	500	0.5	0.1	20.05	4.05	0.50
32	0.05	2	150	50	350	500	5.0	1.0	21.00	41.00	0.50

(PHM hereafter from Panagos et al., 1985)

Comparison of one-stage and two-stage designs (2/2)

Table 3 (*continued*)

Example	β_1	n_x	n_y	h_x	h_y	k_x	k_y	w_x	E(A)
	0.3	5	1	0.34	0.94	3.06	0.01	0.74	14.12
	0.6	5	1	0.34	0.94	3.06	0.01	0.74	14.13
	0.9	4	1	0.12	0.07	3.06	2.16	0.71	16.38
16	0.3	2	3	1.19	1.26	2.68	0.01	1.02	23.99
	0.6	1	7	0.19	0.56	1.68	2.88	0.60	29.43
	0.9	1	5	0.19	0.61	0.01	3.50	0.01	31.74
		3[b]		1.61		2.85			24.59
	0.3	2	1	1.43	1.07	2.63	0.01	0.96	23.19
	0.6	1	2	0.15	0.33	2.41	2.17	0.78	25.42
	0.9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	28.20
18	0.3	17	1	6.80	1.43	3.07	0.01	0.01	34.15
	0.6	10	26	0.37	2.41	2.80	2.20	1.13	37.34
	0.9	1	30	0.09	2.62	0.01	3.73	0.01	38.42
		18[b]		6.54		3.11			34.12
	0.3	17	1	6.69	1.43	3.08	0.01	0.01	34.13
	0.6	10	18	0.28	1.97	2.96	2.03	1.22	36.83
	0.9	8	12	0.20	1.67	2.72	2.45	0.94	37.83
20	0.3	7	2	1.60	0.12	3.83	0.01	0.01	37.56
	0.6	3	9	0.05	0.53	3.35	2.46	1.14	38.41
	0.9	1	9	0.03	0.59	0.01	4.00	0.01	38.54
		7[b]		1.50		3.83			37.57
	0.[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]54
	0.[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]23
	0.9	2	4	0.03	0.39	3.14	2.75	0.77	38.42

E(A) increases either as β_1 increases or as b_3' decreases

The monitoring depends more on the $\bar{Y}$ chart when β_1 gets larger

Assume $\sigma_x = 1,\ \sigma_y = 1,\ a_3' = 0.1a_3,\ a_4' = 0.1a_4$

$b_3' = 0.2b_3$ *or* $b_3' = b_3,\ \beta_1 = 0.3,\ 0.6,\ 0.9$

Comparison of one-stage and two-stage designs (2/2)

Table 3 (*continued*)

The two-stage model always yields a higher E(*A*) than does the PHM model as long as *X* and *Y* are moderately($\beta_1 = 0.6$) and hightly correlated($\beta_1 = 0.9$)

Example	β_1	n_x	n_y	h_x	h_y	k_x	k_y	w_x	E(*A*)
16	0.3	2	3	1.19	1.26	2.68	0.01	1.02	23.99
	0.6	1	7	0.19	0.56	1.68	2.88	0.60	29.43
	0.9	1	5	0.19	0.61	0.01	3.50	0.01	31.74
		3[b]		1.61		2.85			24.59
	0.3	2	1	1.43	1.07	2.63	0.01	0.96	23.19
	0.6	1	2	0.15	0.33	2.41	2.17	0.78	25.42
	0.9	1	2	0.15	0.37	2.06	2.66	0.71	28.20
18	0.3	17	1	6.80	1.43	3.07	0.01	0.01	34.15
	0.6	10	26	0.37	2.41	2.80	2.20	1.13	37.34
	0.9	1	30	0.09	2.62	0.01	3.73	0.01	38.42
		18[b]		6.54		3.11			34.12
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.01	34.13
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1.22	36.83
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.94	37.83
20	0.3	7	2	1.60	0.12	3.83	0.01	0.01	37.56
	0.6	3	9	0.05	0.53	3.35	2.46	1.14	38.41
	0.9	1	9	0.03	0.59	0.01	4.00	0.01	38.54
		7[b]		1.50		3.83			37.57
	0.3	6	1	1.59	0.10	3.71	0.01	0.01	37.54
	0.6	3	5	0.04	0.41	3.56	2.13	1.31	38.23
	0.9	2	4	0.03	0.39	3.14	2.75	0.77	38.42

As β_1 incrases, n_x, h_x, k_x, decrease, and k_y increases

Assume $\sigma_x = 1,\ \sigma_y = 1,\ a_3' = 0.1a_3,\ a_4' = 0.1a_4$

$b_3' = 0.2b_3$ or $b_3' = b_3,\ \beta_1 = 0.3,\ 0.6,\ 0.9$

Sensitivity analysis

Table 2
Cost and process parameters for test examples

Example	λ	c	t_1	t_2	a_1	a_2	a_3	a_4	b_1	b_2	b_3
2	0.01	1	150	50	350	500	5.0	1.0	3.05	4.05	0.05

Table 3
Optimum design for test examples[a]

Example	β_1	n_x	n_y	h_x	h_y	k_x	k_y	w_x	E(A)
2	0.3	15	1	6.20	1.43	2.86	0.01	1.13	134.21
	0.6	9	27	0.25	2.45	2.65	2.2	1.02	137.84
	0.9	1	28	0.09	2.55	0.01	3.56	0.01	138.98
		17[b]		1.54		4.31			134.11

Table 4
The economic penalty associated to the estimation error of λ, c, and β_1

		$E^*(A)$	$E^+(A)$	Penalty (%)
$\lambda_{real} = 0.01$	$\lambda_{estimated}$			
	0.002	146.63	146.12	0.35
	0.005	143.05	142.91	0.10
	0.02	128.53	128.37	0.12
	0.05	106.00	105.27	0.69

Penalty (%) → $1-E^+(A)/E^*(A)$

Conclusions

- 當管制圖應用於製程上，相較於合理性替代變數而言，其運用績效變數去量測為昂貴的，故對監控製程較經濟手法為同時使用其兩者。
- 比較兩階段模式與PHM's一階段模式之數值案例，從期望淨收益角度來看，可確認兩階段對於一階段而言較佳。