Calculating the (Almost) Exact Control Limits for a C-Chart

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Abstract

- Agreement between the exact LCL and UCL, as determined by the lower and upper tail area, is excellent.

- The square-root transformation that stabilizes the variance produces a negatively skewed distribution and tends to give false SPC signals

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- TRANSFORMING THE POISSON DISTRIBUTION TO SYMMETRY
- EXACT INTEGER UCL AND LCL FOR c BAR
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- CONCLUSIONS

Introduction

- The c-chart, used for most Poisson-type data, has a major advantage in introductory statistical process control (SPC) application because of its simplicity

Since the standard deviation for a c-chart is $\sqrt{\bar{c}}$, the usual limits can be calculated quickly as:

$$LCL = \bar{c} - 3\sqrt{\bar{c}} \quad (1)$$

and

$$UCL = \bar{c} + 3\sqrt{\bar{c}} \quad (2)$$

- When the calculated LCL is zero or a negative number, the lower control limit (LCL) does not exist.
- Depending on the value of the mean, there may be numerous false out-of-control (OOC) signals from data exceeding the calculated UCL and/or LCL.

TRANSFORMING THE POISSON DISTRIBUTION TO SYMMETRY

Two equations, based on separate work by the author (Kittlitz, 2003), can be derived so that the (almost) exact 0.00135 (LCL) and the 0.99865 (UCL) limits can be calculated.

$$\mathrm{LCL} = \left[\left(\bar{c} + \frac{1}{12}\right)^{2/3} - 3\left(\frac{2}{3}\right)(\bar{c})^{1/6}\right]^{3/2} + \frac{1}{4} \quad (3)$$

$$\mathrm{UCL} = \left[\left(\bar{c} + \frac{1}{12}\right)^{2/3} + 3\left(\frac{2}{3}\right)(\bar{c})^{1/6}\right]^{3/2} - \frac{3}{4} \quad (4)$$

The corresponding "$\pm 2\sigma$" warning limits, 0.02275 (LWL) and 0.97725 (UWL) can be calculated.

$$\text{LWL} = \left[\left(\bar{c} + \frac{1}{12} \right)^{2/3} - 2\left(\frac{2}{3}\right)(\bar{c})^{1/6} \right]^{3/2} + \frac{1}{4} \qquad (5)$$

and

$$\text{UWL} = \left[\left(\bar{c} + \frac{1}{12} \right)^{2/3} - 2\left(\frac{2}{3}\right)(\bar{c})^{1/6} \right]^{3/2} - \frac{3}{4} \qquad (6)$$

How well do these equations satisfy the exact probability limit?

EXACT INTEGER UCL AND LCL FOR C Bar

- If the LCL calculates to be <0, there is no LCL. The traditional limits are shown in Table 3. The traditional LCL calculation is in error by ~1.9 units, and the traditional UCL calculation is in error by ~0.7 units because of ignoring symmetry.

Table I

Exact control limits at specified $\bar{c}$

$\bar{c}$ for LCL	LCL	$\bar{c}$ for UCL	UCL
6.607675	1	0.052883	1
8.900233	2	0.211682	2
10.869554	3	0.465293	3
12.680501	4	0.791869	4
14.392425	5	1.174966	5
16.034804	6	1.603007	6
17.624837	7	2.067713	7
19.173608	8	2.562994	8
20.688759	9	3.084241	9
22.175823	10	3.627878	10
23.638963	11	4.191070	11
25.081401	12	4.771528	12
26.505695	13	5.367372	13
27.913915	14	5.977038	14
29.307764	15	6.599208	15
30.688663	16	7.232760	16
32.057816	17	7.876730	17
33.416247	18	8.530281	18
34.764841	19	9.192683	19
36.104368	20	9.863293	20
37.435501	21	10.541542	21
38.758835	22	11.226924	22
40.074896	23	11.918987	23
41.384156	24	12.617323	24

Table 2
Comparison of exact and calculated control limits at specified $\bar{c}$

$\bar{c}$ for LCL	LCL	CLCL	$\bar{c}$ for UCL	UCL	CUCL
6.607675	1	1.0	0.052883	1	1.1
8.900233	2	2.0	0.211682	2	2.1
10.869554	3	3.0	0.465293	3	3.0
12.680501	4	4.0	0.791869	4	4.0
14.392425	5	5.0	1.174966	5	5.0
16.034804	6	6.0	1.603007	6	6.0
17.624837	7	7.0	2.067713	7	7.0
19.173608	8	8.0	2.562994	8	8.0
20.688759	9	9.0	3.084241	9	9.0
22.175823	10	10.0	3.627878	10	10.0
23.638963	11	11.0	4.191070	11	11.0
25.081401	12	12.0	4.771528	12	12.0
26.505695	13	13.0	5.367372	13	13.0
27.913915	14	14.0	5.977038	14	14.0
29.307764	15	15.0	6.599208	15	15.0
30.688663	16	16.0	7.232760	16	16.0
32.057816	17	17.0	7.876730	17	17.0
33.416247	18	18.0	8.530281	18	18.0
34.764841	19	19.0	9.192683	19	19.0
36.104368	20	20.0	9.863293	20	20.0
37.435501	21	21.0	10.541542	21	21.0

Note: CLCL—Calculated Lower Control Limit using Eq. (3). CUCL—Calculated Upper Control Limit using Eq. (4).

Table 3
Comparison of exact and traditional control limits at specified $\bar{c}$

$\bar{c}$ for LCL	LCL	TLCL	$\bar{c}$ for UCL	UCL	TUCL
6.607675	1	−1.1	0.052883	1	0.7
8.900233	2	−0.0	0.211682	2	1.6
10.869554	3	1.0	0.465293	3	2.5
12.680501	4	2.0	0.791869	4	3.5
14.392425	5	3.0	1.174966	5	4.4
16.034804	6	4.0	1.603007	6	5.4
17.624837	7	5.0	2.067713	7	6.4
19.173608	8	6.0	2.562994	8	7.4
20.688759	9	7.0	3.084241	9	8.4
22.175823	10	8.0	3.627878	10	9.3
23.638963	11	9.1	4.191070	11	10.3
25.081401	12	10.1	4.771528	12	11.3
26.505695	13	11.1	5.367372	13	12.3
27.913915	14	12.1	5.977038	14	13.3
29.307764	15	13.1	6.599208	15	14.3
30.688663	16	14.1	7.232760	16	15.3
32.057816	17	15.1	7.876730	17	16.3
33.416247	18	16.1	8.530281	18	17.3
34.764841	19	17.1	9.192683	19	18.3
36.104368	20	18.1	9.863293	20	19.3
37.435501	21	19.1	10.541542	21	20.3

Note: TLCC—Traditional Lower Control Limit using Eq. (1). TUCL—Traditional Upper Control Limit using Eq (2).

COMPARISON TO ANOTHER EQUATION

- Ryan和Schwertman 1997提供了兩個回歸方程基礎計算, GREAT應用這些方程，使得下尾面積區 LTA和上尾面積區分別約為 0.00135， ARL平均運行長度約370
- LCL= 1時，C -Chart意味著可以計算出poisson方程為 6.607650687，這樣的LTA= 0.0013500。同樣為UCL = 1時，C -CHART平均計算為0.0528835562使UTA= 0.0013500

- The Ryan and Schwertman equations are an improvetnent over the traditional c-chart calculations as shown by the respective residuals plotted in Figures 1 and 2.

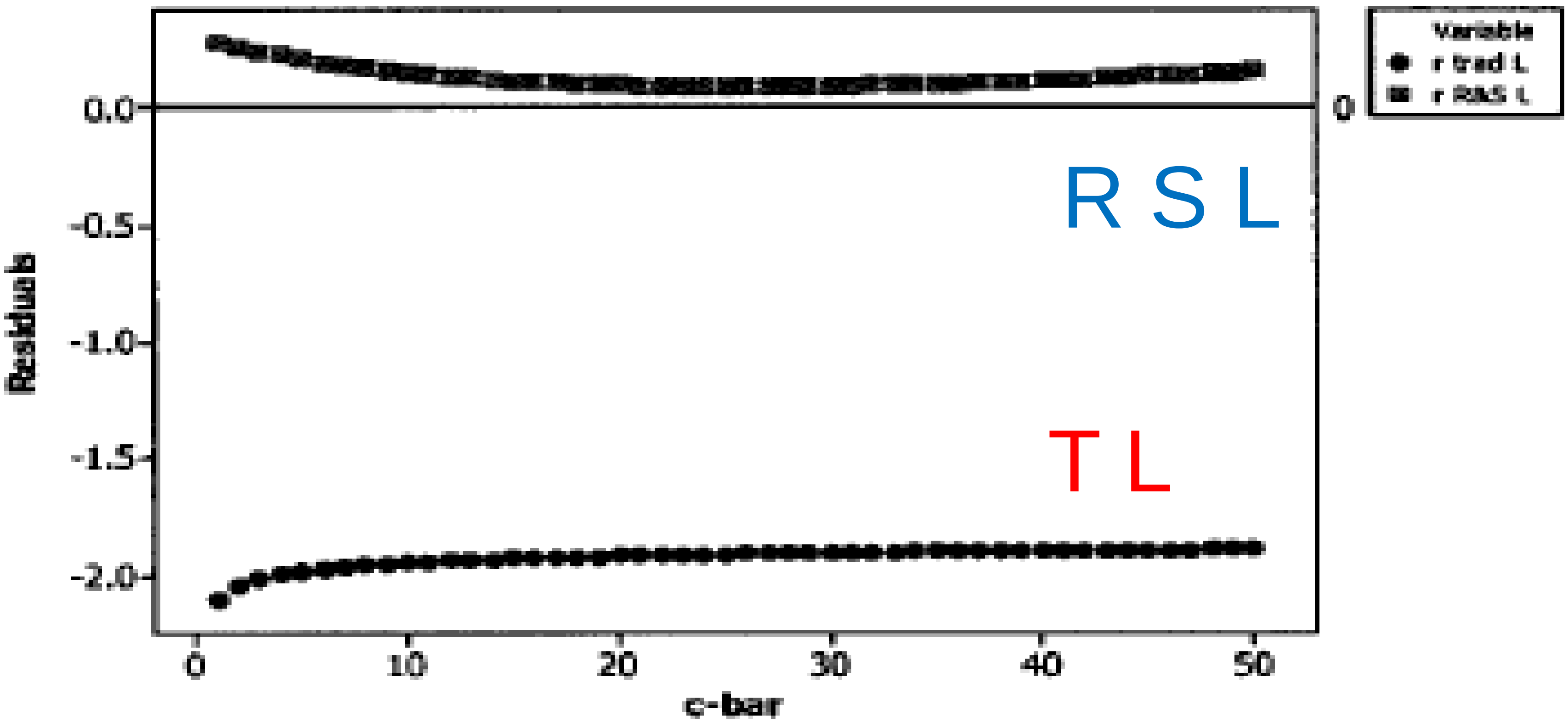

Figure 1. Residuals from traditional LCL and Ryan and Schwertman equation.

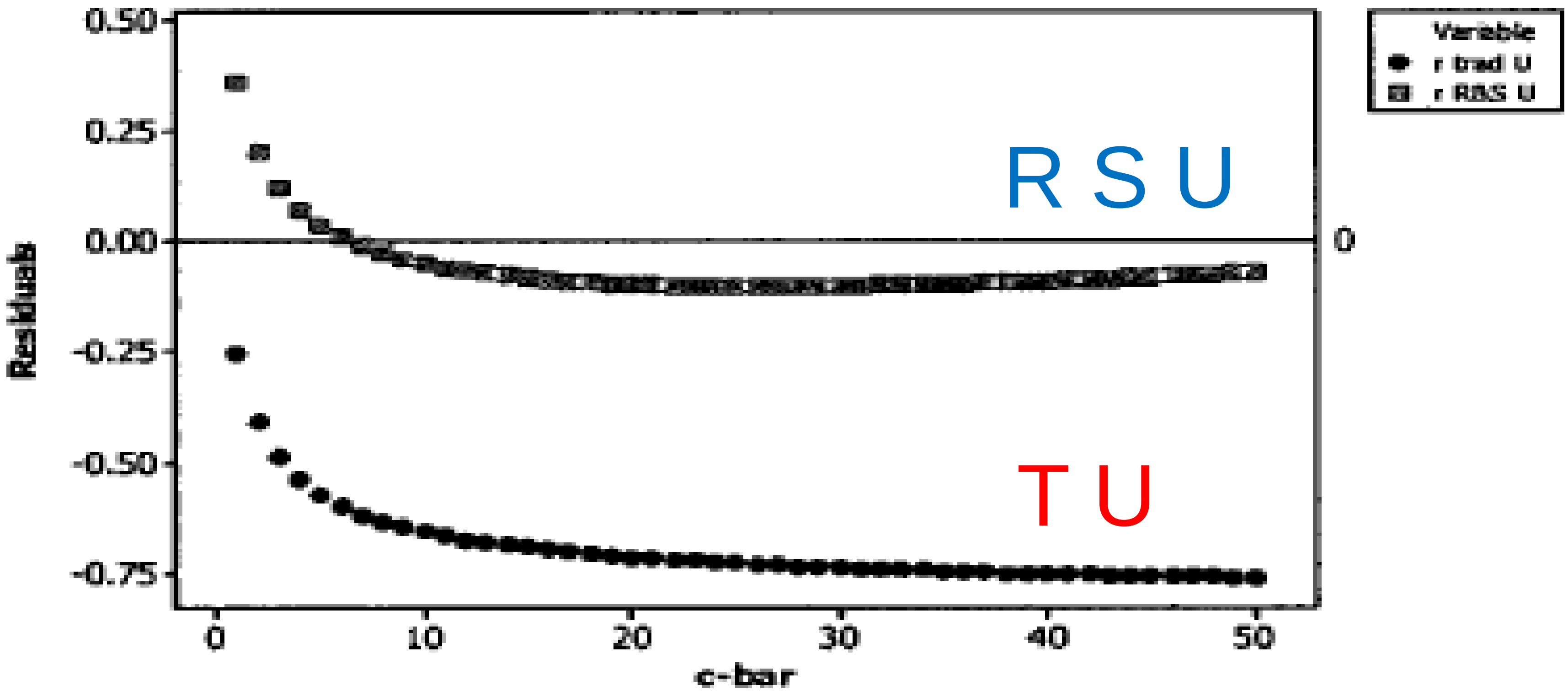

Figure 2. Residuals from traditional UCL and Ryan and Schwertman equation.

PROPOSED LCL AND UCL EQUATIONS FOR POISSON DATA CONTROL CHAR

The proposed equations will use the basic equation

$$c \cong \left[\hat{\mu}_y \pm z\hat{\sigma}_y\right]^{3/2} + k \quad (7)$$

Haldance 1938年發表2/3 -power的會轉換產生一個對稱的Poisson distribution

Read和Cressie 1988年，(p96備註)歸因Anscombe使用一個很小的常數添加到Poisson distribution，再執行2/3 power轉換(Anscombe, 1985)

In completely separate work, the author (Kittlitz, 2003) derived the following equations for symmetry:

$$y = \left(c + \frac{1}{4}\right)^{2/3} \tag{8}$$

for expected mean on the transformed scale,

$$\mu_y \cong \left(\mu + \frac{1}{12}\right)^{2/3} \tag{9}$$

for expected standard deviation on the transformed scale,

$$\sigma_y \cong \left(\frac{2}{3}\right)(\mu)^{1/6} \tag{10}$$

The Poisson equation can be written as:

$$f(x) = \frac{\mu^x e^{-\mu}}{x!} \tag{11}$$

$$x = g(y) = y^{3/2} - \frac{1}{4} \tag{12}$$

- 隨著建議的轉換Eq(8)用X代替C，我們可以獲得以下幾點：

- Brownlee (1967, p. 45).
  - 我們假設為 Y = F(X) {x是一個嚴格單調函數的X，X =g{y}是一個嚴格單調的函數在離散情況下Let *p y{y}* be the probability function of *Y.* 只要x需要一個特定的值x，它與它的概率*Px{x},* so *px{x}* =Px{G(y)}=PY{y} (應用於(11) and (12))

$$g(y) = \Pr(Y = y) = \Pr(X = y^{3/2} - 1/4)$$
$$= \frac{\mu^{y^{3/2}-1/4} e^{-\mu}}{(y^{3/2} - 1/4)} \quad (13)$$

To obtain the $E(Y)$, we have to sum the series as:

$$E(Y) = \sum y e^{-\mu} \frac{\mu^{y^{3/2}-1/4}}{(y^{3/2} - 1/4)} \quad (14)$$

- For u= 5, the *E{Y)* = 2.95529, 標準差 0.87305, 偏態-0.00886, 峰態3.02319標準化第五次後 -0.05278,標準化第六次15.09179

(Gaussian distribution)

- Equations (9) and (10) give excellent results as shown in Table 4.

- Since Eqs. (9) and (10) are simple and give very good results, they will be used to calculate the control limits shown previously and repeated next.

Calculating the (Almost) Exact Control Limits for a c-Chart

Table 4

Comparison on transformed scale of expected and calculated mean and of expected and calculated standard deviation

$\bar{c}$	Expected mean	Calculated mean	Expected std dev	Calculated std dev
1	1.074772	1.054811	0.613366	0.656667
2	1.632477	1.631195	0.737387	0.748308
3	2.117022	2.118428	0.799770	0.800625
4	2.553271	2.554719	0.841128	0.839947
5	2.955286	2.956417	0.873054	0.871774
6	3.331576	3.332430	0.899686	0.898671
7	3.687632	3.688290	0.922814	0.922058
8	4.027206	4.027730	0.943373	0.942809
9	4.352987	4.353416	0.961930	0.961500
10	4.666979	4.667340	0.978871	0.978533
11	4.970727	4.971036	0.994473	0.994201
12	5.265451	5.265721	1.008949	1.008724
13	5.552138	5.552377	1.022460	1.022271
14	5.831601	5.831814	1.035137	1.034976
15	6.104516	6.104708	1.047085	1.046945
16	6.371458	6.371632	1.058390	1.058267
17	6.632918	6.633078	1.069122	1.069014
18	6.889321	6.889468	1.079343	1.079247
19	7.141036	7.141172	1.089102	1.089016
20	7.388390	7.388516	1.098444	1.098366

$$\text{LCL} = \left[\left(\mu + \frac{1}{12}\right)^{2/3} - 3\left(\frac{2}{3}\right)(\mu)^{1/6}\right]^{3/2} + \frac{1}{4} \qquad (15)$$

$$\text{UCL} = \left[\left(\mu + \frac{1}{12}\right)^{2/3} + 3\left(\frac{2}{3}\right)(\mu)^{1/6}\right]^{3/2} - \frac{3}{4} \qquad (16)$$

- 殘差圖 3和圖4顯示顯著改善Eqs. (15) and (16)及如何計算的方程。(和Ryan和Schwertman方程相比)

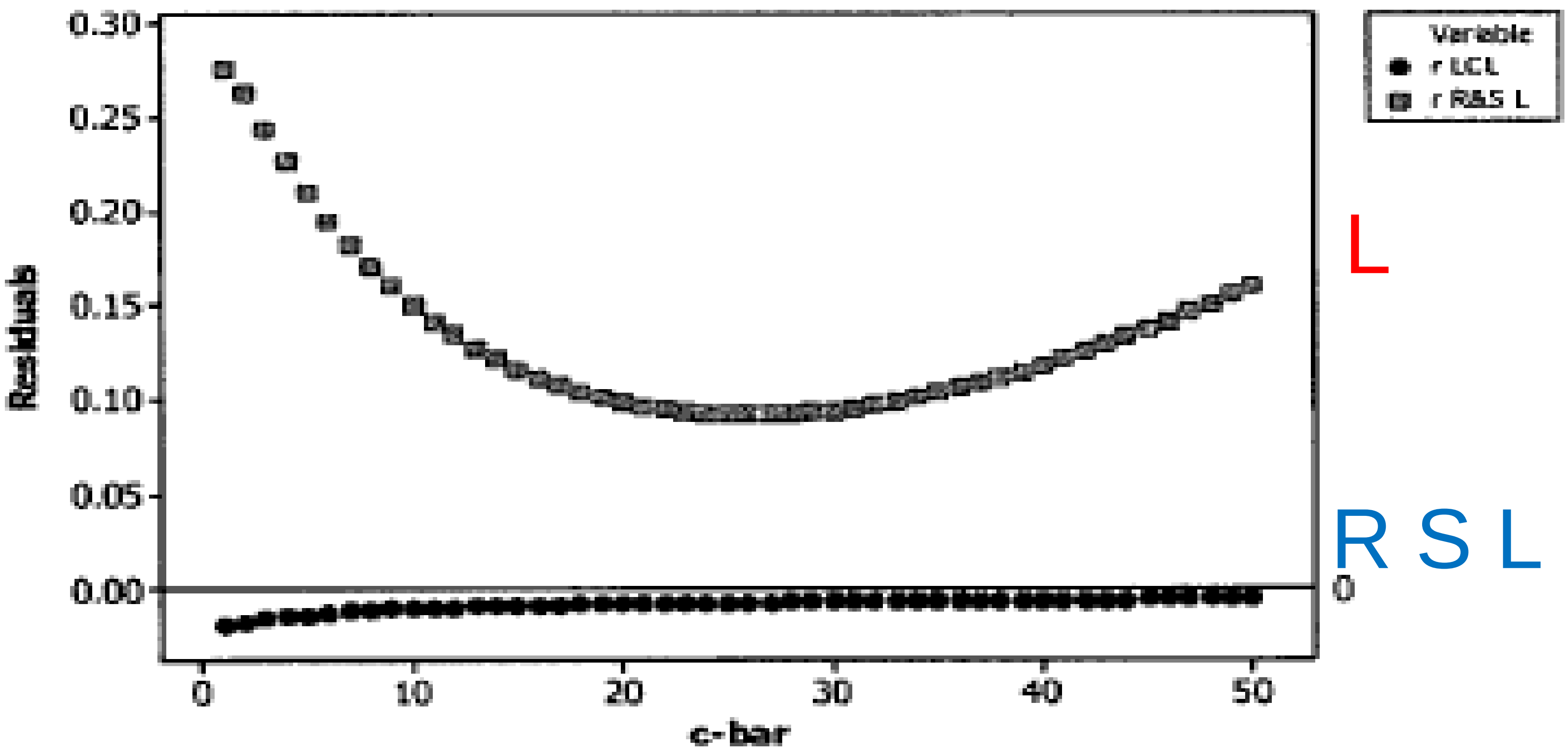

Figure 3. Residuals from proposed LCL and Ryan and Schwertman equation.

$$\mathrm{LCL} = \left[\left(\mu + \frac{1}{12}\right)^{2/3} - 3\left(\frac{2}{3}\right)(\mu)^{1/6}\right]^{3/2} + \frac{1}{4} \qquad (15)$$

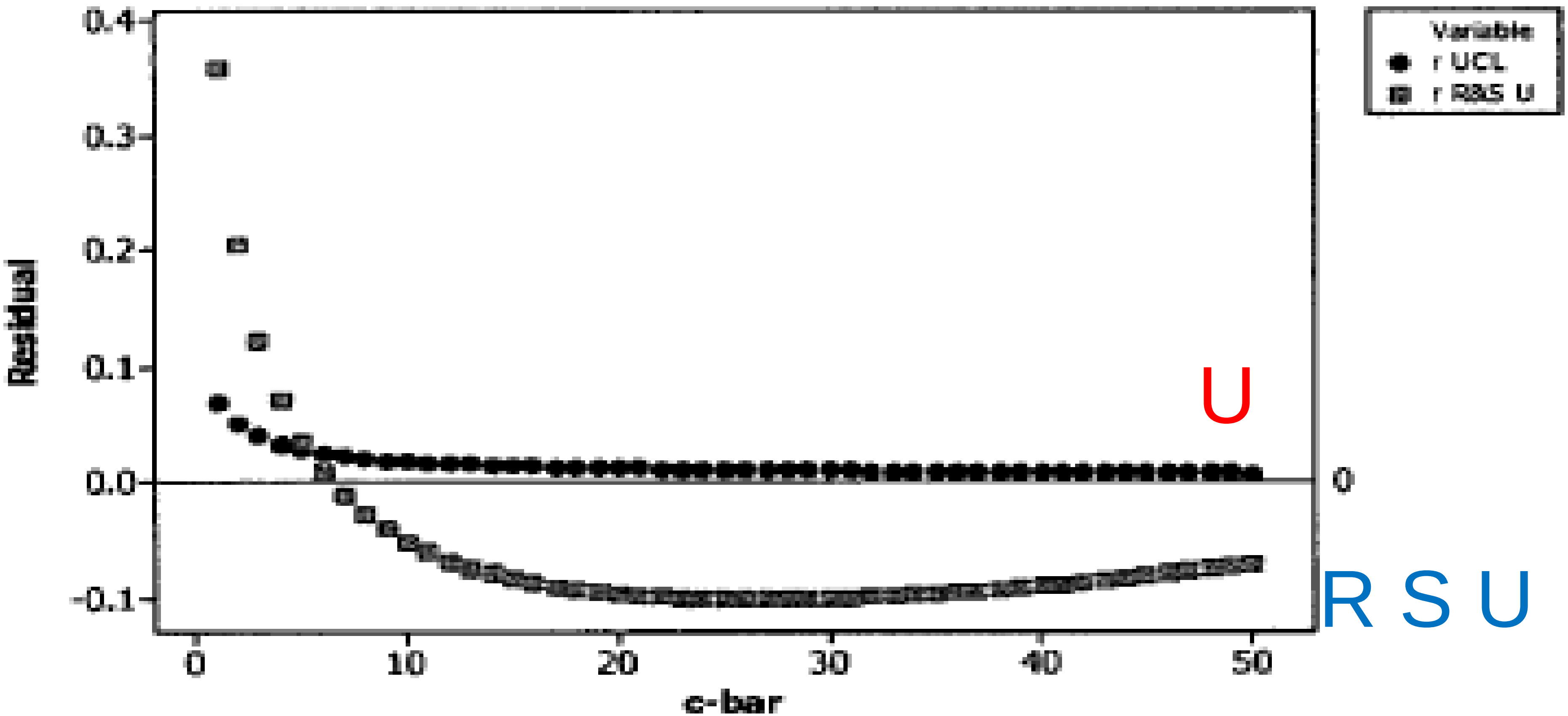

Figure 4. Residuals from proposed UCL and Ryan and Schwertman equation.

$$\mathrm{UCL} = \left[\left(\mu + \frac{1}{12}\right)^{2/3} + 3\left(\frac{2}{3}\right)(\mu)^{1/6}\right]^{3/2} - \frac{3}{4} \qquad (16)$$

PROPOSED LWL AND UWL EOR POISSON DATA CONTROL CHART

- These limits are set at "$\pm 2a$ about the mean" or equal tail area of 0.02275.

- The proposed equations were shown previously as Eqs. (5) and (6) and give excellent results also. A summary table is shown in Table 5.

Table 5

Comparison of exact and calculated warning limits at specified $\bar{c}$

$\bar{c}$ for LWL	LWL	CLWL	$\bar{c}$ for UWL	UWL	CUWL
3.783187	1	1.0	0.230144	1	1.1
5.682711	2	2.0	0.596287	2	2.1
7.348434	3	3.0	1.057965	3	3.1
8.901677	4	4.0	1.582871	4	4.0
10.385377	5	5.0	2.153234	5	5.0
11.820604	6	6.0	2.758208	6	6.0
13.219549	7	7.0	3.390641	7	7.0
14.590016	8	8.0	4.045533	8	8.0
15.937356	9	9.0	4.719231	9	9.0
17.265429	10	10.0	5.408973	10	10.0
18.577126	11	11.0	6.112610	11	11.0
19.874681	12	12.0	6.828434	12	12.0
21.159862	13	13.0	7.555057	13	13.0
22.434095	14	14.0	8.291339	14	14.0
23.698553	15	15.0	9.036325	15	15.0
24.954214	16	16.0	9.789207	16	16.0
26.201900	17	17.0	10.549296	17	17.0

Note: CLWL—Calculated Lower Warning Limit using Eq. (5) and CUWL—Calculated Upper Warning Limit using Eq. (6).

$$\text{LWL} = \left[\left(\bar{c} + \frac{1}{12}\right)^{2/3} - 2\left(\frac{2}{3}\right)(\bar{c})^{1/6}\right]^{3/2} + \frac{1}{4} \quad (5)$$

and

$$\text{UWL} = \left[\left(\bar{c} + \frac{1}{12}\right)^{2/3} - 2\left(\frac{2}{3}\right)(\bar{c})^{1/6}\right]^{3/2} - \frac{3}{4} \quad (6)$$

AN EXAMPLE TO ILLUSTRATE THE CALCULATIONS

- Ryan (2000, p. 172) 提出了一套25點，將不符合的數據來說明這些計算。不符合的數據有一個平均的7.56;範圍為 1到17

  (假設，數據是Poisson distribution)。
- 建議使用EQ 15和EQ16的計算方程。

$$\text{LCL} = \left[\left(\bar{c} + \frac{1}{12}\right)^{2/3} - 3\left(\frac{2}{3}\right)(\bar{c})^{1/6}\right]^{3/2} + \frac{1}{4}$$

$$= \left[\left(7.56 + \frac{1}{12}\right)^{2/3} - 2(7.56)^{1/6}\right]^{3/2} + \frac{1}{4} = 1.37$$

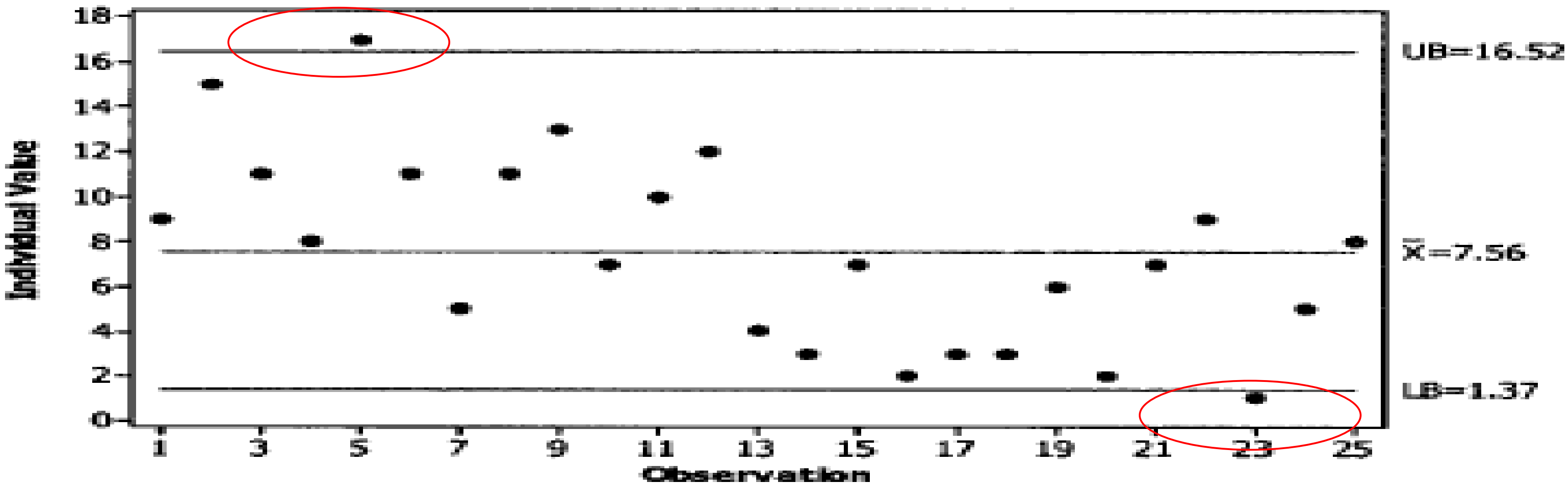

Figure 5. Nonconformity data chart using proposed LCL and UCL.

$$\mathrm{UCL} = \left[\left(\bar{c} + \frac{1}{12}\right)^{2/3} + 3\left(\frac{2}{3}\right)(\bar{c})^{1/6}\right]^{3/2} - \frac{3}{4}$$

$$= \left[\left(7.56 + \frac{1}{12}\right)^{2/3} + 2(7.56)^{1/6}\right]^{3/2} - \frac{3}{4} = 16.92$$

- Figure 5 is a plot of the data, which shows two "outof-control" signals.
- An EWMA and/or CUSUM analysis could be applied to refine the conclusions about the two populations means and when the apparent decrease occurred
- Another possible explanation for this data set is that a general decrease in time has occurred, but that discussion is beyond the scope of this paper.

CONCLUSIONS

- The agreement between the exact LCL and UCL, as determined by the lower tail area and upper tail area, is excellent.

- Similar equations are proposed for calculating the LWL and UWL for Poisson type data.

END