Monitoring Multi-Attribute Processes Based on NORTA Inverse Transformed Vectors

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1. Introduction and Literature Review

- Statistical control charts, in general, consist of the variable and attribute control charts, for which researchers have developed various methodologies.
- Almost all researchers have focused on the first category of control charting and only a few methods have been proposed to monitor multi-attribute processes.

1. Introduction and Literature Review

- Patel (1973) proposed a Hotelling-type T^2 chart to monitor observations from multivariate distributions in which the marginals are Binomial or Poisson.
- Wu et al. (2006) proposed an algorithm for the optimization design of the *np* control chart.
- Larpkiattaworn (2003) proposed a back propagation neural network (*BPNN*) for two-attribute processes for the case of bivariate Binomial and bivariate Poisson.

1. Introduction and Literature Review

- Skinner et al. (2006) proposed a new statistic, called deleted-Y , to be computed for each variable.
- Niaki and Abbasi (2007b) proposed a skewness reduction approach and by simulation experiments showed their approach performs better than the other competing methods.

1. Introduction and Literature Review

- In this article, we propose a T^2 control chart, based on the Patel's (1973) method, to monitor the number of defects in multi-attribute processes.
- we first propose a data transformation technique and then employ the T^2 control chart for the transformed data.
- The goal is not just to detect process deteriorations but to monitor process improvements.

2. The Proposed Normalizing Transformation

- There are two approaches to reduce skewness in univariate attribute control charts: (1) adding some correction value to the control limits based on the value of the skewness; and (2) applying a normalizing transformation technique
- most researchers prefer to use the second approach.

2. The Proposed Normalizing Transformation

- some transformation, such as the square root, inverse, arcsin, Q-transformation , and

  parabolic inverse have been proposed in the literature.
- Xie et al. (2000) proposed double square root transformation for Geometric distribution.
- Based on the NORmal-To-Anything (NORTA) method, we propose an inverse transformation technique to transform multi-attribute data to a shape close to a multivariate normal distribution.

2. The Proposed Normalizing Transformation

- The goal of the NORTA algorithm is to generate a k-dimensional random vector ***X*** with the following properties:

  - $X_i \sim F_{X_i}, i = 1, 2, \ldots, k$, where each F_{X_i} is an arbitrary cumulative distribution function (cdf); and
  - $\mathrm{Corr}[\mathbf{X}] = \Sigma_{\mathbf{X}}$, where $\Sigma_{\mathbf{X}}$ is given.

- the vector **X** by a transformation of a k-dimensional standard multivariate normal (MVN) vector $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_k)^T$

$$\mathbf{X} = \begin{pmatrix} F_{X_1}^{-1}[\Phi(z_1)] \\ F_{X_2}^{-1}[\Phi(z_2)] \\ \vdots \\ F_{X_k}^{-1}[\Phi(z_k)] \end{pmatrix} \qquad (1)$$

where Φ is the cdf of a univariate standard normal and $F_X^{-1}(u) \equiv \inf\{x : F_X(x) \geq u\}$ denotes the inverse cdf.

2. The Proposed Normalizing Transformation

$$\rho_X(i,j) = \mathrm{Corr}[X_i, X_j] = \mathrm{Corr}\{F_{X_i}^{-1}[\Phi(z_i)], F_{X_j}^{-1}[\Phi(z_j)]\} \quad i \neq j. \tag{2}$$

Since

$$\mathrm{Corr}(X_i, X_j) = \frac{E(X_i X_j) - E(X_i)E(X_j)}{\sqrt{Var(X_i)Var(X_j)}} \tag{3}$$

$$\begin{aligned} E(X_i X_j) &= E\{F_{X_i}^{-1}[\Phi(z_i)] F_{X_j}^{-1}[\Phi(z_j)]\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{X_i}^{-1}[\Phi(z_i)] F_{X_j}^{-1}[\Phi(z_j)] \varphi_{\rho_z(i,j)}(z_i, z_j)\, dz_i\, dz_j \end{aligned} \tag{4}$$

where $\varphi_{\rho_z(i,j)}(z_i, z_j)$ is the standard bivariate normal probability density function (pdf) with correlation $\rho_Z(i,j)$.

2. The Proposed Normalizing Transformation

- In order to generate a k-dimensional random vector by the NORTA algorithm we need to solve Eq. (4) for each pair of the variables.
- Cario and Nelson (1997) presented some theorems that describe the properties of Σ_Z which are helpful in solving Eq. (4).
- in this research, we propose an inverse transformation formula for transforming a vector of multi-attribute variables to new variables with approximate multi-variate normal distribution.

2. The Proposed Normalizing Transformation

$$\mathbf{Y} = [Y_1, Y_2, \ldots, Y_k]^T = [\Phi^{-1}(F_{X_1}(x_1)), \Phi^{-1}(F_{X_2}(x_2)), \ldots, \Phi^{-1}(F_{X_k}(x_k))]^T. \quad (5)$$

assuming marginal *Poisson* distributions and using Eq. (5), we transform the original vector **X (1)**to the new one **Y(5)** and estimate the correlation matrix of the transformed vector $\mathbf{S_Y}$ by the method of moments.

3. T^2 Multi-Attribute Control Charts Based on Transformed Data

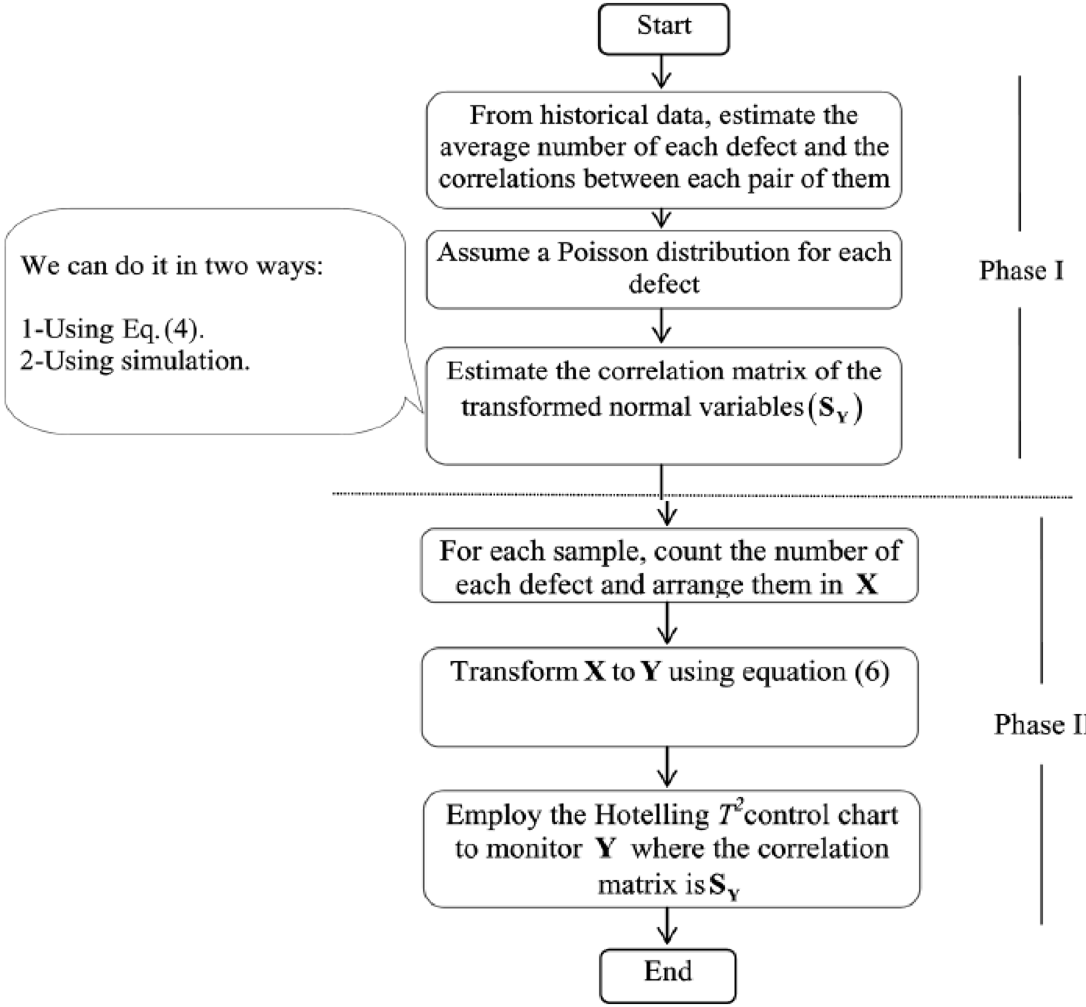
- we first transform the vector of the original quality attributes to a new attribute vector with approximate multivariate normal distribution using Eq. (5).
- we determine the control limits for our multivariate control chart. If the plotted points

$$T^2 = (\mathbf{Y} - \overline{\mathbf{Y}})^T \mathbf{S}_{\mathbf{Y}}^{-1} (\mathbf{Y} - \overline{\mathbf{Y}}) \text{ (where } \mathbf{Y} = (Y_1, Y_2, \ldots, Y_k)^T)$$

fall within the control limits, the process is in control.

Otherwise, it is out of control.

3. *T*2 Multi-Attribute Control Charts Based on Transformed Data



3. *T2* Multi-Attribute Control Charts Based on Transformed Data

3.1 *Numerical Examples*

- Examples 1, 2 and 3 contain 中 *Poisson* parameters with medium, 大 large, and small sizes, 小 respectively.

Table 1
General statistics of the original and the transformed vectors

	Data	Mean and Covariance	Skewness 1st variable	Skewness 2nd variable	Kurtosis 1st variable	Kurtosis 2nd variable	*P*-Value (JB test) 1st variable	*P*-Value (JB test) 2nd variable
#1	Original data	$\hat{\mu}_X = [3.940, 4.954]^T$ $\widehat{\mathrm{Cov}}(X) = \begin{pmatrix} 4.022 & 1.902 \\ 1.902 & 4.969 \end{pmatrix}$	0.531	0.423	3.327	3.278	< 0.00001	< 0.00001
	Transformed data	$\hat{\mu}_Y = [0.225, 0.207]^T$ $\widehat{\mathrm{Cov}}(Y) = \begin{pmatrix} 0.939 & 0.401 \\ 0.401 & 0.946 \end{pmatrix}$	0.079	0.023	2.952	2.897	0.056	0.259
#2	Original data	$\hat{\mu}_X = [8.067, 8.067]^T$ $\widehat{\mathrm{Cov}}(X) = \begin{pmatrix} 7.815 & 4.104 \\ 4.104 & 8.177 \end{pmatrix}$	0.356	0.363	3.129	3.214	< 0.00001	< 0.00001
	Transformed data	$\hat{\mu}_Y = [0.203, 0.201]^T$ $\widehat{\mathrm{Cov}}(Y) = \begin{pmatrix} 0.939 & 0.490 \\ 0.490 & 0.982 \end{pmatrix}$	0.019	0.030	3.016	2.960	0.835	0.578
#3	Original data	$\hat{\mu}_X = [2.992, 3.967]^T$ $\widehat{\mathrm{Cov}}(X) = \begin{pmatrix} 2.985 & 0.975 \\ 0.975 & 3.963 \end{pmatrix}$	0.6102	0.4789	3.3760	3.1574	< 0.00001	< 0.0001
	Transformed data	$\hat{\mu}_Y = [0.294, 0.240]^T$ $\widehat{\mathrm{Cov}}(Y) = \begin{pmatrix} 0.889 & 0.258 \\ 0.258 & 0.925 \end{pmatrix}$	0.148	0.055	2.861	2.854	< 0.0001	0.0297

3. *T*2 Multi-Attribute Control Charts Based on Transformed Data

- For a true normal distribution, the sample skewness should be near zero and the sample kurtosis should be near three.
- the transformed variables follow approximate normal distributions. It means that the proposed normalizing transformation method works well for *Poisson* distributions with medium- and large-size parameters.

4. Simulation Experiments

- Skinner et al. (2006) used Johnson et al.'s (1997) method to generate data from a joint Poisson distribution in which the joint probability distribution is given by Eq. 6:

$$f_{\mathbf{X}}(\mathbf{x}) = \exp(-(\lambda_0 + \lambda_1 + \lambda_2)) \frac{\lambda_1^{x_1}}{x_1!} \frac{\lambda_2^{x_2}}{x_2!} \sum_{i=0}^{\min(x_i)} \binom{x_1}{i} \binom{x_2}{i} i! \left(\frac{\lambda_0}{\lambda_1 \lambda_2} \right)^i. \qquad (6)$$

where λ_1 and λ_2 are the parameters of the marginal distribution and λ_0 is the covariance. We note that the mean of the first and the second marginal distribution

4. Simulation Experiments

- ## 4.1. *Simulation Experiment* 1

estimate the parameters of the marginal distributions for each defect type as $\lambda_1 = 3$ and $\lambda_2 = 2$ with covariance being two. To monitor both attributes simultaneously, we first use simulation to generate 5,000 data sets for a distribution with the above-normal vector using Eq. (5). Based on the proposed transformation method we have:

$$\hat{\mu}_Y = [0.2238, 0.2523]^T, \quad \text{and} \quad \widehat{\mathrm{Cov}}(\mathbf{Y}) = \begin{pmatrix} 0.9525 & 0.4083 \\ 0.4083 & 0.9286 \end{pmatrix}.$$

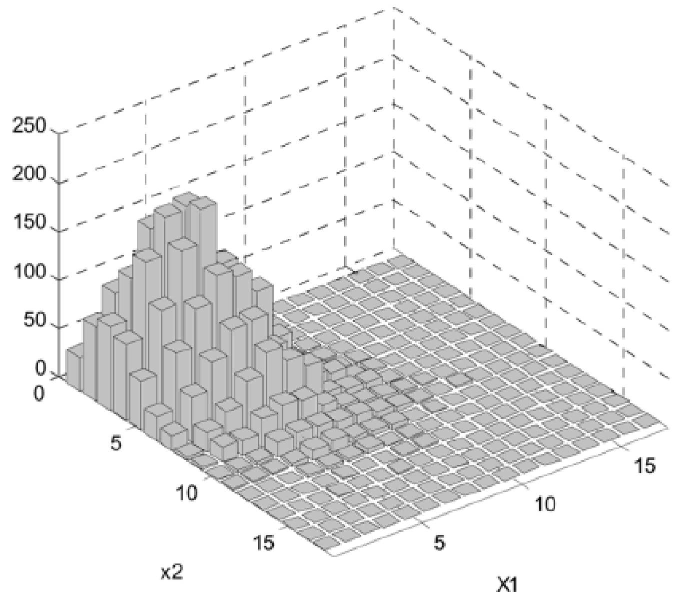


Figure 2. The joint probability distribution of the original vector in simulation 1.

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4. Simulation Experiments

- **4.1. *Simulation Experiment* 1**

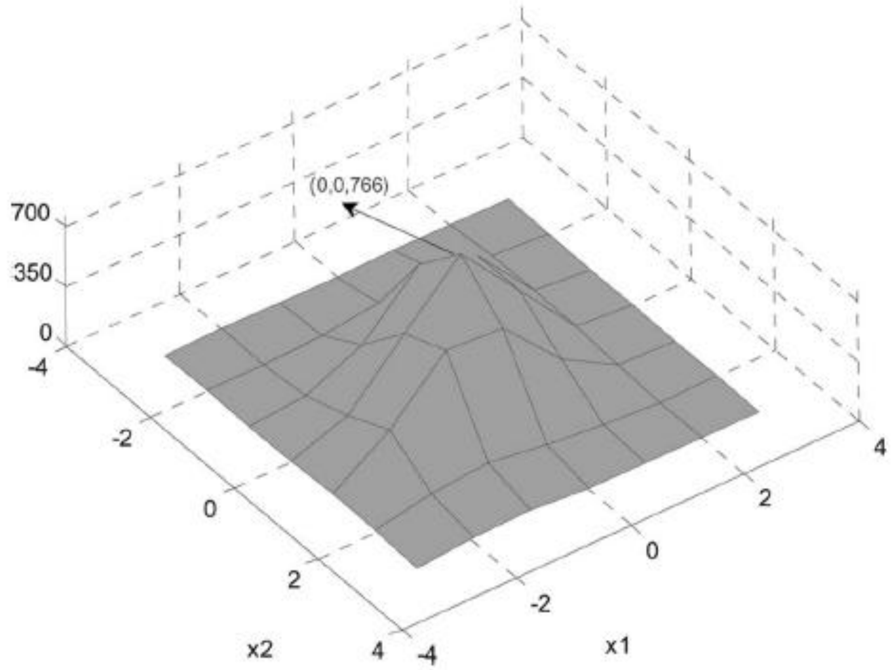

Figure 3. The joint probability distribution of the transformed vector in simulation 1.

4. Simulation Experiments

4.2. *Simulation Experiment 2*

given production process. Assuming the number of defects of three types follow a multivariate Poisson distribution, we estimate the parameters on an in-control process as $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 3$ with the estimated variance–covariance matrix of $\hat{\Sigma} = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 1 \end{pmatrix}$.

To monitor all three attributes simultaneously, first we generate 5,000 observations for a vector from the above distribution. Then we perform the NORTA inverse transformation to arrive at the following estimates:

$$\hat{\mu}_{\mathbf{Y}} = [0.2768, 0.2555, 0.2307]^T \quad \text{and} \quad \widehat{\text{Cov}}(\mathbf{Y}) = \begin{pmatrix} 0.9244 & 0.4694 & 0.2130 \\ 0.4694 & 0.9139 & 0.2085 \\ 0.2130 & 0.2085 & 0.9609 \end{pmatrix}.$$

Table 2

ARL_1 values for different Shifts in simulation experiment 1

	$ARL_1\left(\frac{Mean}{Std}\right)$					
Shift→	$(\sigma_1, 0)$	$(0, \sigma_2)$	(σ_1, σ_2)	$(2\sigma_1, 0)$	$(0, 2\sigma_2)$	$(2\sigma_1, 2\sigma_2)$
Proposed	24.128	23.828	20.325	5.206	5.222	4.439
method	23.828	24.027	20.829	4.726	4.815	4.128
Deleted-Y	19.322	63.429	67.278	4.248	9.957	29.645
method	18.38	65.273	64.57	3.646	9.2753	29.483
Shift→	$(3\sigma_1, 0)$	$(0, 3\sigma_2)$	$(3\sigma_1, 3\sigma_2)$	$(0.5\sigma_1, -0.5\sigma_2)$	$(-0.5\sigma_1, 0.5\sigma_2)$	$(-\sigma_1, \sigma_2)$
Proposed	2.263	2.303	1.903	48.110	49.595	8.685
method	1.719	1.782	1.370	48.932	50.272	8.324
Deleted-Y	1.9675	3.339	17.204	30.007	93.953	12.169
method	1.3268	2.7625	16.904	29.723	90.772	12.267

4. Simulation Experiments

- 4.2. *Simulation Experiment 2*

Table 3

ARL_1 values for different shifts in simulation experiment 2

	$ARL_1\left(\frac{Mean}{Std}\right)$						
Shift→	$(\sigma_1, 0, 0)$	$(0, \sigma_2, 0)$	$(0, 0, \sigma_3)$	$(\sigma_1, \sigma_2, 0)$	$(\sigma_1, 0, \sigma_3)$	$(0, \sigma_2, \sigma_3)$	$(\sigma_1, \sigma_2, \sigma_3)$
Proposed	27.402	25.283	38.784	23.030	17.030	15.592	16.608
method	26.304	25.055	39.007	23.366	16.325	15.189	16.311
Deleted-*Y*	28.011	27.658	35.096	36.902	35.016	35.182	58.608
method	26.027	26.105	34.931	36.001	34.582	34.271	57.010
	$2(\sigma_1, 0, 0)$	$2(0, \sigma_2, 0)$	$2(0, 0, \sigma_3)$	$2(\sigma_1, \sigma_2, 0)$	$2(\sigma_1, 0, \sigma_3)$	$2(0, \sigma_2, \sigma_3)$	$2(\sigma_1, \sigma_2, \sigma_3)$
Proposed	5.825	5.573	8.613	5.117	3.377	3.207	3.479
method	5.1716	5.129	8.3424	4.5361	2.8165	2.6558	2.955
Deleted-*Y*	5.438	5.479	6.987	9.623	9.518	9.392	26.107
method	5.025	5.193	6.466	9.190	8.913	8.741	25.312
	$3(\sigma_1, 0, 0)$	$3(0, \sigma_2, 0)$	$3(0, 0, \sigma_3)$	$3(\sigma_1, \sigma_2, 0)$	$3(\sigma_1, 0, \sigma_3)$	$3(0, \sigma_2, \sigma_3)$	$3(\sigma_1, \sigma_2, \sigma_3)$
Proposed	2.473	2.427	3.310	2.105	1.530	1.483	1.532
method	1.8999	1.7949	2.7395	1.5083	0.88844	0.88758	0.92707
Deleted-*Y*	2.220	2.233	2.720	4.151	4.122	4.202	15.127
method	1.939	2.109	2.170	3.826	3.692	3.857	14.442
	$0.4(-\sigma_1, \sigma_2, 0)$	$0.4(\sigma_1, -\sigma_2, 0)$	$0.4(-\sigma_1, 0, \sigma_3)$	$0.4(0, \sigma_2, -\sigma_3)$	$0.4(0, -\sigma_2, \sigma_3)$	$0.4(\sigma_1, 0, -\sigma_3)$	$(0.4\sigma_1, 0.4\sigma_2, -\sigma_3)$
Proposed	71.671	81.687	108.904	99.063	118.405	104.902	44.685
method	72.618	82.532	109.06	95.994	116.86	102.31	43.324
Deleted-*Y*	80.400	81.104	101.190	83.191	99.223	82.696	39.244
method	78.983	80.923	100.023	82.390	97.891	81.474	38.836

4. Simulation Experiments

4.3. *Sensitivity Analysis on the Parameter Values*

Table 5 summarizes the results of the simulation experiment.

The results of Table 5 show that while the proposed method performs well in situation in which the parameters posses small values, in majority of mean-shift cases it outperforms the deleted-Y procedure.

Table 5

ARL_1 values for different shifts in small-valued parameters

	$ARL_1\left(\frac{Mean}{Std}\right)$					
Shift→	$(\sigma_1, 0)$	$(0, \sigma_2)$	(σ_1, σ_2)	$(2\sigma_1, 0)$	$(0, 2\sigma_2)$	$(2\sigma_1, 2\sigma_2)$
Proposed	9.061	9.520	9.007	2.682	2.764	2.494
method	8.437	8.887	9.193	2.330	2.502	2.302
Deleted-Y	12.088	13.355	43.397	2.749	2.941	18.867
method	11.604	12.614	43.027	2.197	2.415	18.202
	$(3\sigma_1, 0)$	$(0, 3\sigma_2)$	$(3\sigma_1, 3\sigma_2)$	$(0.5\sigma_1, -0.5\sigma_2)$	$(-0.1\sigma_1, 0.5\sigma_2)$	$(-0.1\sigma_1, \sigma_2)$
Proposed	1.678	1.788	1.587	18.503	29.054	8.861
method	1.064	0.958	1.009	17.942	28.680	8.313
Deleted-Y	1.518	1.495	11.281	13.307	18.597	3.475
method	0.895	0.862	10.777	12.801	18.045	2.891

4. Simulation Experiments

- **4.3. *Sensitivity Analysis on the Parameter Values***

Table 6

ARL_1 values for different shifts in large-valued parameters

	$ARL_1\left(\frac{Mean}{Std}\right)$					
Shift→	$(\sigma_1, 0)$	$(0, \sigma_2)$	(σ_1, σ_2)	$(2\sigma_1, 0)$	$(0, 2\sigma_2)$	$(2\sigma_1, 2\sigma_2)$
Proposed	34.613	33.837	23.442	6.694	6.573	4.015
method	33.824	34.011	23.162	6.200	6.080	3.480
Deleted-Y	51.856	23.955	77.288	10.007	5.204	39.193
method	51.461	23.790	76.422	9.473	4.665	38.842
	$(3\sigma_1, 0)$	$(0, 3\sigma_2)$	$(3\sigma_1, 3\sigma_2)$	$(0.5\sigma_1, -0.5\sigma_2)$	$(-0.5\sigma_1, 0.5\sigma_2)$	$(-\sigma_1, \sigma_2)$
Proposed	2.498	2.395	1.581	61.022	56.611	11.628
method	1.918	1.832	0.948	60.688	57.252	11.155
Deleted-Y	3.571	2.136	23.552	60.924	33.634	6.949
method	3.011	1.526	22.715	60.089	33.009	6.536

5. Conclusion and Recommendations for Future Research

- In this article, we first proposed a new transformation technique to approximate the skewed distribution to a joint probability distribution in which the marginals are normal, and then applied a multivariate control charting technique on the transformed data.
- When we use original data the ARL_0 value is very low and hence the method is not applicable.

5. Conclusion and Recommendations for Future Research

- However, when we transform the data, the ARL_0 will have a more appropriate value.
- Furthermore, by simulation we showed that the proposed method performs better than the deleted-Y method in most of the mean-shift scenarios.
- after the transformation phase, instead of T^2 control chart we may want to examine other multivariate control charting techniques such as MEWMA and MCUSUM as well.

THE END