Transformation of the bathtub failure rate data in reliability for using Weibull-model analysis

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Abstract

- A new model based on the use of data transformation is presented for modeling bathtub-shaped hazard rates.
- Parameter estimation methods are studied for this new (transformation) approach.

Introduction

- The term transformation appears in statistical literature in many different contexts.
- Another area of statistical application is lifetime data analysis.
- In order to deal with problems indicating bathtub-shaped failure rates.

The power transformation

- A typical set of such data is formalized as n pairs of i.i.d. r.v. $(X_i, \delta_i), \quad i = 1, 2, \cdots, n$

$$X_i = \min(T_i, C_i) \quad \text{and} \quad \delta_i = \begin{cases} 1, & \text{if } T_i \le C_i, \text{ (uncensored)} \\ 0, & \text{if } C_i < T_i, \text{ (censored)}, \end{cases} \tag{1}$$

T_i is the lifetime, C_i the censoring time and δ_i the censoring indicator,

The power transformation

- Suppose also that there exists parameter θ in the transformation given in (2) such that, the transformed data $(y_i, \delta_i),\quad i = 1, 2, \cdots, n$ are distributed according to Weibull (σ, α) model.

- We propose use of transformation $g(x, \theta)$ given by

$$g(x,\theta) = \left(\frac{x}{1-\theta x}\right), \qquad \max(x) < 1/\theta, \quad \theta \geq 0. \qquad (2)$$

$$x = \frac{y}{1+\theta y}, \qquad (3)$$

The power transformation

The survival function of the original observations corresponding to the transformation $g(x, \theta)$ is given by,

$$s(x) = P(X > x) = 1 - F(x) = \exp[-(\frac{x}{\sigma})^{\alpha}]$$

$$\begin{aligned} S(x) &= P(X > x) \\ &= P\left\{\frac{y}{1+\theta y} > x\right\}, \quad \text{(substitution for } x \text{ from (3))} \\ &= \exp\left[-\left\{\frac{1}{\sigma}\left(\frac{x}{1-\theta x}\right)\right\}^{\alpha}\right], \quad \text{since } y \sim \text{Weibull}(\sigma, \alpha),\ 0 < x < 1/\theta. \end{aligned} \qquad (4)$$

The density function, $f(x)$ is then easily obtained by differentiation of $1 - S(x)$ and the hazard function $h(x)$ is given as $f(x)/(S(x))$. These are given, respectively, by:

$$f(x) = \frac{\alpha}{\sigma}\left\{\frac{1}{\sigma}\left(\frac{x}{1-\theta x}\right)\right\}^{\alpha-1} \exp\left[-\left\{\frac{1}{\sigma}\left(\frac{x}{1-\theta x}\right)\right\}^{\alpha}\right]\frac{1}{(1-\theta x)^2}, \qquad (5)$$

and

$$h(x) = \frac{\alpha}{\sigma^{\alpha}}\left(\frac{x}{1-\theta x}\right)^{\alpha+1}\frac{1}{x^2}, \quad \sigma > 0,\ \alpha > 0,\ 0 < x < 1/\theta. \qquad (6)$$

The power transformation

$$h(x) = \frac{\alpha}{\sigma^{\alpha}} \left(\frac{x}{1-\theta x} \right)^{\alpha+1} \frac{1}{x^2}.$$

Hazard function shape	Transformation $g(x)$
Increasing	$\alpha > 1$
Bathtub	$0 < \alpha < 1$

$$\log h(x) = (\alpha + 1) \log \left(\frac{x}{1-\theta x} \right) - \log x^2 + C,$$

$$\frac{d}{dx}(\log h(x)) = \frac{\alpha+1}{x(1-\theta x)} - \frac{2}{x}$$

$$= \frac{\alpha - 1 + 2\theta x}{x(1-\theta x)}.$$

$$x^* = \frac{1-\alpha}{2\theta}$$

$$\frac{d^2}{dx^2}(\log h(x)) = \frac{2\theta}{x(1-\theta x)} + (\alpha - 1 + 2\theta x)\frac{2\theta x - 1}{x^2(1-\theta x)^2}.$$

Statistical inference

- Maximum likelihood estimation

$$\mathcal{L}(\alpha,\sigma,\theta) = \prod_{i=1}^{n} \{h(y_i)\}^{\delta_i} S(y_i)$$

$$= \prod_{i=1}^{n} \left\{\frac{\alpha}{\sigma}\left(\frac{y_i}{\sigma}\right)^{\alpha-1}\right\}^{\delta_i} \exp\left(-\left(\frac{y_i}{\sigma}\right)^{\alpha}\right),$$

$$\mathcal{L}(\alpha,\sigma,\theta;x) = \prod_{i=1}^{n} \left\{\frac{\alpha}{\sigma}\left(\frac{y_i}{\sigma}\right)^{\alpha-1}\right\}^{\delta_i} \exp\left(-\left(\frac{y_i}{\sigma}\right)^{\alpha}\right) |\mathbf{J}(\theta;\mathbf{x}_i)|, \quad |J(\theta;x_i)| = \frac{1}{(1-\theta x)^2}$$

$$\ell(\alpha,\sigma,\theta) = \log(\mathcal{L}(\alpha,\sigma,\theta;x))$$

$$= d\log\left(\frac{\alpha}{\sigma}\right) + (\alpha-1)\sum_{i=1}^{n}\delta_i \log\left(\frac{y_{1i}}{\sigma}\right) - \sum_{i=1}^{n}\left(\frac{y_{1i}}{\sigma}\right)^{\alpha} + \sum_{i=1}^{n}\log|\mathbf{J}(\theta;\mathbf{x}_i)|,$$

Statistical inference

- Estimates of Weibull parameters

$$\begin{cases} \breve{\sigma} = \exp\left[\frac{1}{n}\sum_{i=1}^{n}\left(1.6655\frac{(i-5)}{(n-5)} + 0.16724\right)\log y_{i:n}\right], \\ \breve{\alpha} = \frac{1}{0.69313n}\sum_{i=1}^{n}\left(2\frac{(i-0.5)}{(n-1)} - 1\right)\log y_{i:n} \end{cases} \quad (10)$$

where $y_{1:n} \le y_{2:n} \le \cdots y_{n:n}$ denote the order statistics of a complete sample from a Weibull distribution.

$$F(x) = 1 - S(x) = 1 - \exp\left[-\left\{\frac{1}{\sigma}\left(\frac{x}{1-(x/\phi)}\right)\right\}^{\alpha}\right], \quad x < \phi, \qquad \phi = \theta^{-1}$$

$$\ell(\alpha, \sigma, \tilde{\phi}) = d\log\left(\frac{\alpha}{\sigma}\right) + (\alpha - 1)\sum_{i=1}^{n}\delta_i \log\left(\frac{y_i}{\sigma}\right) - \sum_{i=1}^{n}\left(\frac{y_i}{\sigma}\right)^{\alpha} + \sum_{i}^{n}\log|\mathrm{J}(\tilde{\phi}; \mathrm{x}_i)|,$$

where $y_i = x/(1-(x/\phi))$ and $\mathrm{J}(\tilde{\phi}; \mathrm{x}_i) = 1/(1-(x/\phi))^2$.

$$\tilde{\phi} = X_{n:n}$$

Examples

- Lifetimes of 50 devices, Aarset [1]

0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

- The Akaike Information Criteria(AIC)

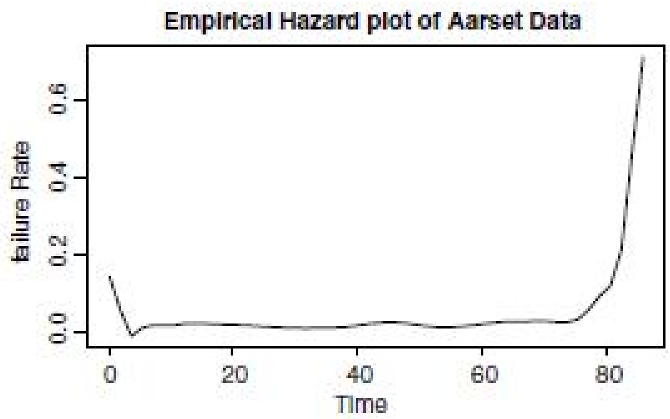
$$\text{AIC} = -2 \times \log(\text{maximumlikelihood}) + 2 \times (\text{number of parameters fitted}).$$

The estimated parameters and AIC values for different models for Aarset's data.

Model	Estimated parameters	AIC	Rank
Transformation model	$\hat{\sigma} = 354.4160,\ \hat{\alpha} = 0.3850,\ \hat{\theta} = 0.0116$	402.42	1
Generalized Weibull [26]	$\hat{\sigma} = 410.0360,\ \hat{\alpha} = 2.2990,\ \hat{\lambda} = 1.9727$	407.60	2
Additive Burr XII [36]	$c_1 = 0.5067,\ s_1 = 2137,\ k_1 = 5.5$ $c_2 = 152.93,\ s_2 = 85.2526,\ k_2 = 0.5$	444.63	3
Haupt and Schabe [14]	$t_0 = 128\ \beta = 0.09$	470.52	4
Min Xie et al. [39]	$\alpha = 110.09,\ \beta = 0.8408,\ \lambda = 0.0141$	478.49	5
The additive model [20]	$a = 0,\ b = 30.069,\ c = 0.0912,\ d = 0.4996$	532.89	6

Examples

- Hazard plots of Aarset's Data



- Inference about the transformation parameter θ

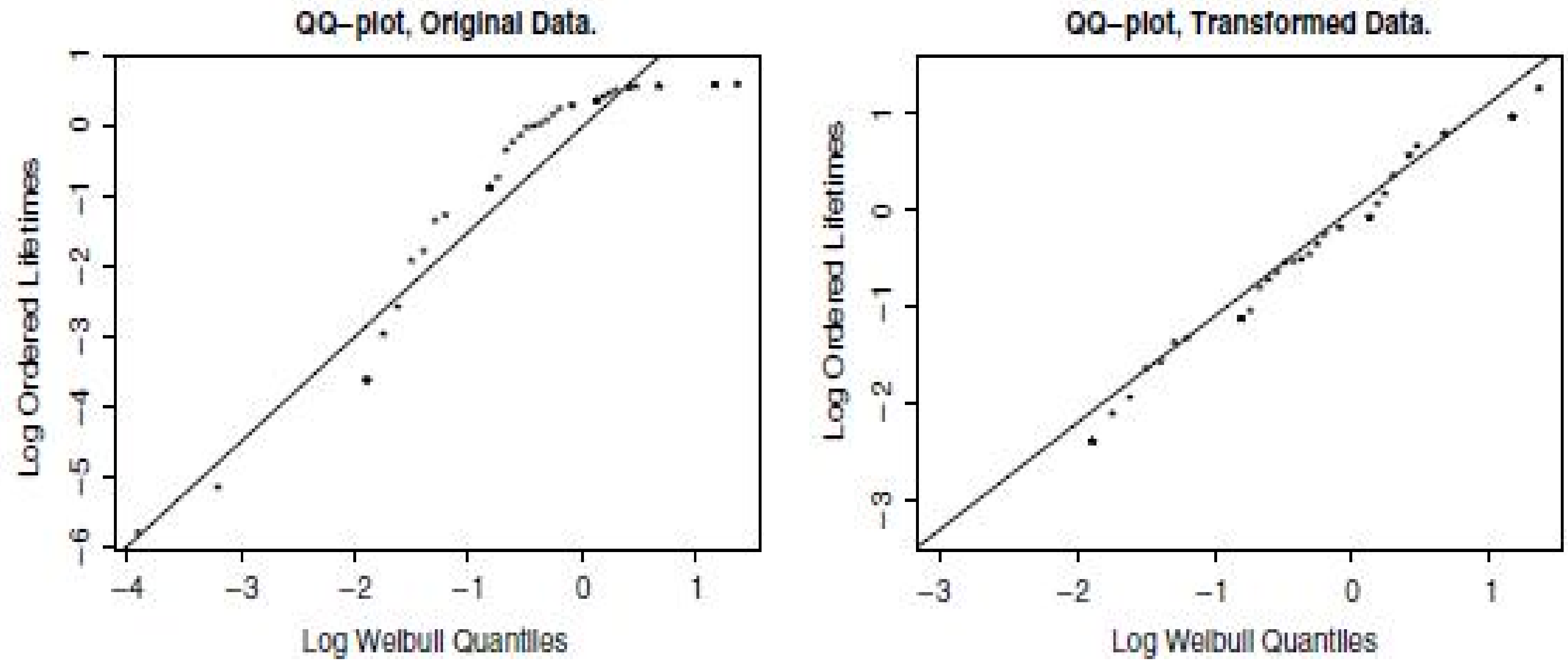
$$H_0 : \theta = 0, \quad v.s. \quad H_1 : \theta \neq 0$$

$$\because 95\% CI = (0.01141, 0.01163)$$

$$\therefore 0 \notin (0.01141, 0.01163)$$

Examples

- Weibull goodness-of-fit test



$$P - value < 0.0001$$

Examples

- The competing model fits appear in Fig. 3.

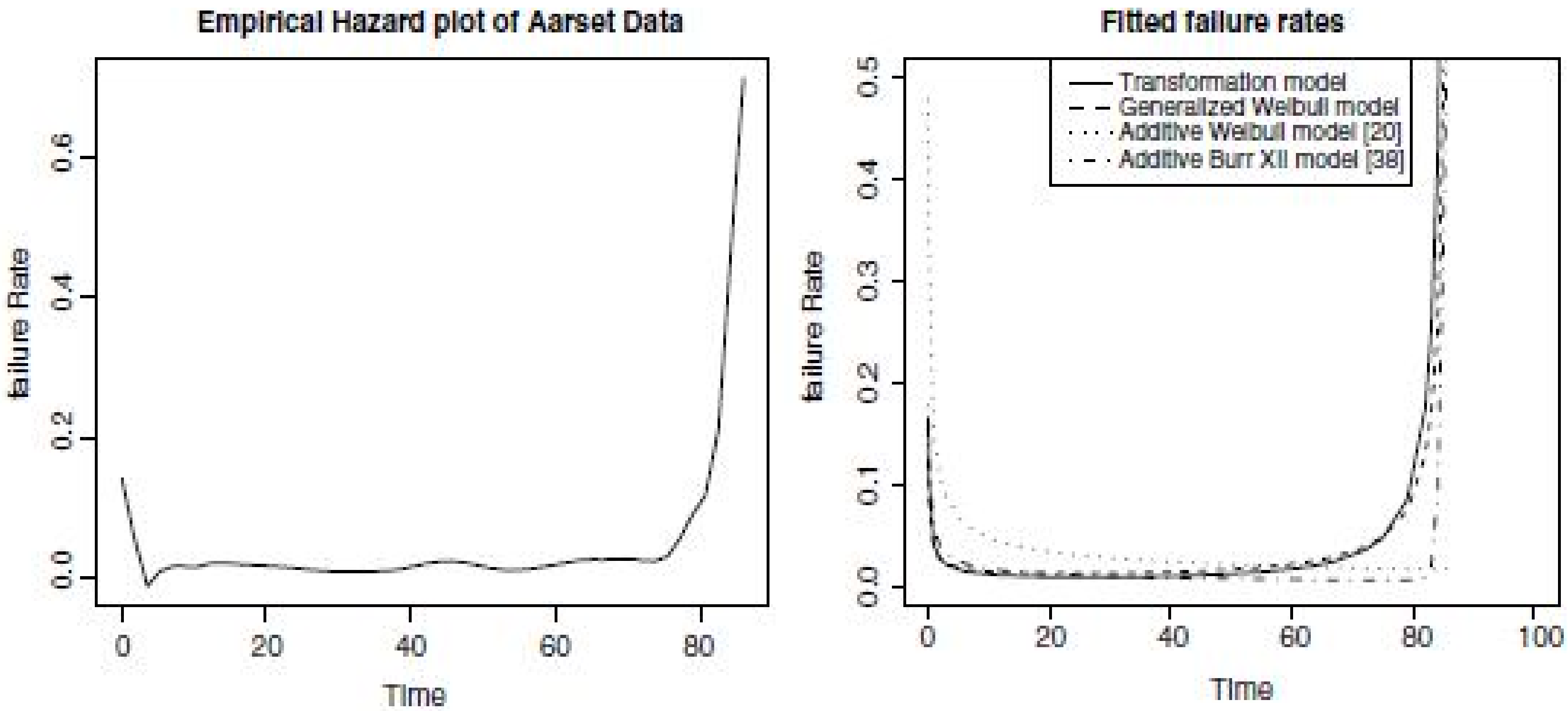


Fig. 3. Comparison of the empirical hazard rate and the hazard rates of four model fits for Aarset data.

Conclusions

- We have presented a data-transformation approach for analyzing lifetime data with bathtub-shaped failure rate.
- The new method serves as a good alternative for modeling bathtub-shaped failure rate data.
- THE END THANKS