Should Exponentially Weighted Moving Average and Cumulative Sum Charts Be Used With Shewhart Limits?

Marion R. Reynolds, Jr.

Departments of Statistics and Forestry
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061
(*mrr@vt.edu*)

Zachary G. Stoumbos

Department of Management Science and Information Systems
Rutgers Business School
Rutgers University
Piscataway, NJ 08854
(*stoumbos@andromeda.rutgers.edu*)

Traditional Shewhart control charts are usually considered effective for detecting large shifts in process parameters, but ineffective for detecting small shifts. For detecting small parameter shifts, it is much better to use exponentially weighted moving average (EWMA) control charts or cumulative sum (CUSUM) control charts, but these charts are not considered as effective as Shewhart charts for large parameter shifts. It is frequently recommended that EWMA or CUSUM charts be used in combination with a Shewhart chart to gain the benefits of both types of charts, so that both small and large shifts can be detected quickly. Here we consider the problem of process monitoring when a continuous process variable is being observed and the objective is to detect small or large shifts in either the process mean μ or the process standard deviation σ. In this situation it is customary to use a combination of two control charts, one chart designed to monitor μ, and the other designed to monitor σ. For this situation, the best EWMA or CUSUM chart for monitoring μ is based on sample means, and the best chart for monitoring σ is based on squared deviations from target. An EWMA or CUSUM chart combination based on sample means and squared deviations from target is very effective for detecting small or large shifts in μ or σ. We show that this type of combination is more effective in terms of overall performance than other combinations that do not include the chart based on squared deviations from target and generally are at least as effective as any of the combinations that include the Shewhart chart. Thus we conclude that it is not really necessary to use a Shewhart chart with an EWMA or CUSUM chart to obtain the best overall performance, but it is necessary to use the EWMA or CUSUM chart based on squared deviations from target.

KEY WORDS: Average time to signal; Monitoring; Process mean; Process variance; Robust control chart; Statistical process control; Steady state.

1. INTRODUCTION

Control charts are used to monitor a process to detect special causes that produce changes in the process output. In traditional practice, Shewhart charts with "three-sigma" control limits are usually used. Shewhart control charts are usually considered effective for detecting large shifts in process parameters, but not very effective for detecting small or moderate shifts. Exponentially weighted moving average (EWMA) charts and cumulative sum (CUSUM) charts, in contrast, are considered effective for detecting small and moderate shifts, but not as effective as Shewhart charts for detecting large shifts.

In an effort to obtain the best features of each type of control chart, a Shewhart chart can be used in combination with an EWMA or CUSUM chart. This type of combination is frequently referred to as adding Shewhart limits to the EWMA or CUSUM chart and also has been called a combined Shewhart–CUSUM or Shewhart–EWMA scheme. These control chart combinations have been studied in the statistical process control (SPC) literature and are frequently recommended for use in practical applications (see, e.g., Lucas 1982; Klein 1996; Hawkins and Olwell 1998; Ryan 2000; Woodall 2000; Montgomery 2005). For example, the use of a combined Shewhart–CUSUM scheme in a chemical manufacturing application was described by Westgard, Groth, Aronsson, and de Verdier (1977), and in a nonmanufacturing biomedical application by Blacksell, Glefson, Lunt, and Chamnpood (1994). In general, the overall performance of the combination of a Shewhart chart and an EWMA or CUSUM chart is quite good compared with the overall performance of the Shewhart chart alone or the EWMA or CUSUM chart alone.

When the process variable of interest, say X, is continuous, it is usually assumed that the distribution of this process variable is normal, and then the effect of a special cause is to change the process mean μ and/or the process standard deviation σ. The standard practice is to use two control charts in combination, with one chart designed to detect changes in μ and the other designed to detect changes in σ.

The best EWMA or CUSUM charts for detecting changes in μ are based on sample means, and the best EWMA or CUSUM charts for detecting changes in σ are based on squared deviations from target. Recent research on combinations of an EWMA or CUSUM sample-means chart and squared-deviations chart has shown that such combinations perform extremely well (see Reynolds and Stoumbos 2004a, b). It turns out that the squared-deviations chart, although designed for detecting shifts in σ, is also very effective for detecting large changes in μ. Thus the squared-deviations chart serves essentially the

TECHNOMETRICS, NOVEMBER 2005, VOL. 47, NO. 4
DOI 10.1198/004017005000000382

same function as the addition of the Shewhart chart, while providing much better ability to detect shifts in σ. The main purpose of this article is to show that the combination of an EWMA or CUSUM sample-means chart and squared-deviations chart performs so well that in most cases the addition of Shewhart limits is not needed for monitoring μ and σ.

The specific content of this article is an investigation of the effect of adding a Shewhart chart for μ to EWMA charts that are designed for monitoring both μ and σ. Numerical results are not included here for CUSUM charts, because CUSUM and EWMA charts have very similar behavior (see, e.g., Reynolds and Stoumbos 2004a), and thus the conclusions obtained here for EWMA charts carry over directly to CUSUM charts.

2. SAMPLING FROM THE PROCESS

The process is said to be in control when μ is at its in-control or *target* value, μ_0, and σ is at its in-control value, σ_0. It is assumed that process monitoring will continue over a long period, and the objective is to detect any special cause that changes μ from μ_0 and/or increases σ above σ_0. In some applications, detecting a decrease in σ may also be an objective, but here we focus on the problem of detecting increases in σ, because this will be sufficient to address the issue of the Shewhart limits considered here. In most applications, σ_0 (and sometimes μ_0) must be estimated during a preliminary phase, and here we assumed that these in-control parameter values have been estimated accurately enough so that they can be treated as known quantities for future monitoring.

We consider two types of shifts in μ or σ produced by special causes. The first type of shift, called a *sustained* shift, is assumed to persist until it is detected by the control chart and action is taken to eliminate the special cause. The second type of shift, a *transient* shift, is the result of a special cause that affects the process for only a relatively short period (see Reynolds and Stoumbos 2004b). Let l represent the duration of the transient shift. This means that when the transient shift occurs, the parameter remains at the shifted value for l time periods, then returns to the in-control value even if there is no signal by a control chart. Most of the work in the SPC literature on the properties of control charts assumes that a shift is a sustained shift. We also concentrate on sustained shifts here, but also include some results for transient shifts, because one justification sometimes given for using Shewhart charts is that they are effective for detecting transient shifts. It will be convenient to express the size of shifts in μ and σ in terms of $\delta = |\mu - \mu_0|/\sigma_0$ and $\psi = \sigma/\sigma_0$. We assume that it is desirable to maintain tight control of the process, so that it is important to detect both small and large changes in these process parameters.

Suppose that a sample of $n \geq 1$ observations is taken from the process every d time units. Let $\mathbf{X}_k = (X_{k1}, X_{k2}, \ldots, X_{kn})$ represent the sample available at the kth sampling point, and, when $n > 1$, let $\bar{X}_k$ and S_k^2 represent the sample mean and variance. It is assumed that all observations within and between samples are independent. If $n > 1$, then it is assumed that the n observations in the sample are taken at essentially the same time, so the time between the individual observations in a sample is small enough to be neglected.

3. DEFINITIONS OF THE CONTROL CHARTS

The Shewhart $\bar{X}$ chart for monitoring μ signals at sample k if $\bar{X}_k$ falls outside of the control limits,

$$\mu_0 \pm h_X \frac{\sigma_0}{\sqrt{n}}.$$

When $n = 1$, the Shewhart $\bar{X}$ chart reduces to the Shewhart X chart.

The EWMA chart for detecting changes in μ is based on the control statistic

$$E_k^X = (1 - \lambda)E_{k-1}^X + \lambda \bar{X}_k, \qquad k = 1, 2, \ldots,$$

where λ is a smoothing or tuning parameter satisfying $0 < \lambda \leq 1$, the starting value is $E_0^X = \mu_0$, and the control limits are

$$\mu_0 \pm h_{EX} \sqrt{\frac{\lambda}{n(2 - \lambda)}} \sigma_0.$$

The control limits are based on $\sqrt{\lambda/(n(2-\lambda))}\sigma_0$, the asymptotic in-control standard deviation of E_k^X. The superscript "X" in E_k^X is used here to distinguish this EWMA statistic from other statistics based on S^2 or X^2, as defined later. The EWMA control chart was originally proposed by Roberts (1959), and since then, a number of authors have investigated this chart (see, e.g., Crowder 1987; Lucas and Saccucci 1990; Yashchin 1993). Herein the EWMA chart based on the statistic E_k^X is referred to as the EWMA$_X$ chart, and similar notation is used for charts based on S^2 and X^2.

When $n > 1$, an EWMA chart for monitoring σ can be based on the sample variance S_k^2. Detecting increases in σ is the primary concern here, so we use a one-sided control chart with EWMA statistic

$$E_k^{S^2} = (1 - \lambda) \max\{E_{k-1}^{S^2}, \sigma_0^2\} + \lambda S_k^2, \qquad k = 1, 2, \ldots,$$

where $E_0^{S^2} = \sigma_0^2$ and the upper control limit (UCL) is

$$\sigma_0^2 + h_{ES^2} \sqrt{\frac{2\lambda}{(n-1)(2-\lambda)}} \sigma_0^2.$$

Crowder and Hamilton (1992), Gan (1995), and Morais and Pacheco (2000) investigated EWMA charts based on sample variances, but they used $\ln S_k^2$ instead of S_k^2 in the EWMA.

Control charts based on *squared deviations from target* have been investigated from the perspective of using only one control chart to monitor both μ and σ (see, e.g., Reynolds and Ghosh 1981; Domangue and Patch 1991; Shamma and Amin 1993). Here we view such charts primarily as charts for monitoring σ (see also MacGregor and Harris 1993), with the very helpful property that they also effectively detect large shifts in μ. Consider a one-sided control chart with EWMA statistic

$$E_k^{X^2} = (1 - \lambda) \max\{E_{k-1}^{X^2}, \sigma_0^2\} + \lambda \sum_{i=1}^{n} \frac{(X_{ki} - \mu_0)^2}{n}, \qquad k = 1, 2, \ldots,$$

where $E_0^{X^2} = \sigma_0^2$ and the UCL is

$$\sigma_0^2 + h_{EX^2} \sqrt{\frac{2\lambda}{n(2-\lambda)}} \sigma_0^2.$$

Here the superscript/subscript "X^2" is used to represent $\sum_{i=1}^{n}(X_{ki}-\mu_0)^2$. The EWMA_{X^2} chart has several important advantages compared with the EWMA_{S^2} chart. The EWMA_{X^2} chart has an extra degree of freedom, it can be used when $n=1$, and it plays an important role in detecting large mean shifts when used in combination with the EWMA_X chart. Reynolds and Stoumbos (2001a, b; 2004a), Stoumbos and Reynolds (2000), and Stoumbos, Reynolds, and Woodall (2003) recently investigated the properties of the EWMA_{X^2} chart used in combination with the EWMA_X chart for monitoring μ and σ.

Herein we investigate EWMA charts based on $n=1$ or 4, and λ must be chosen carefully to make the results comparable. If λ_n represents the value of λ used when the sample size is n, then we choose λ_1 for $n=1$ and λ_n for $n>1$, so that the sum of the weights for a set of n individual observations equals the weight of a sample mean when samples of $n>1$ are taken. This requires that

$$\lambda_n = 1-(1-\lambda_1)^n \qquad \text{or} \qquad \lambda_1 = 1-(1-\lambda_n)^{1/n}, \qquad (1)$$

so these equations provide a method for determining λ_n from λ_1, and vice versa, for the EWMA charts considered here. For example, if $n=4$ and $\lambda_4=.1$, then the corresponding value of λ_1 for $n=1$ is $\lambda_1=1-(1-\lambda_4)^{1/4}=.02600$ (see Reynolds and Stoumbos 2004a; Reynolds and Kim 2005 for more discussion of this issue). When a combination of two EWMA charts is used for monitoring μ and σ, these two charts may not use the same value of λ, so we use λ_μ to represent the λ for the EWMA chart for μ and λ_σ to represent the λ for the EWMA chart for σ (which can be either the EWMA_{S^2} chart or the EWMA_{X^2} chart).

Each of the EWMA charts defined here for monitoring μ and σ has a CUSUM chart analog. For example, Reynolds and Stoumbos (2004a, b) have provided definitions of the CUSUM charts based on sample means and squared deviations from target.

4. CHART PARAMETER SELECTION AND PERFORMANCE MEASURES

When comparing control charts in this article, we always use $n/d=1.0$ and refer to the time unit as an hour. The average time to signal (ATS) of a control chart is defined as the expected length of time from the start of process monitoring until a signal is generated. The false-alarm rate used here when comparing control charts corresponds to an in-control ATS of 1,481.6 hours, which is the in-control ATS of the Shewhart $\bar{X}$ chart, when $n=4$, $d=4.0$ hours, the observations are normal, and the control limits are determined using $h_X=3$. The conclusions about the control chart performance that we obtain depend on n, so we consider $n=4$ with $d=4.0$ and $n=1$ with $d=1.0$.

When two or more control charts are used in combination, the individual in-control ATS values must be above 1,481.6 to give a joint in-control ATS of 1,481.6. For most of the comparisons in this article, the control limits of each chart in a combination were adjusted so that the charts have equal individual in-control ATS values. For example, for the combination of the EWMA_X and EWMA_{X^2} charts, the control limits of the charts were adjusted so that each chart has the same individual in-control ATS (around 2,700) when the observations are normal. But for the EWMA_X and Shewhart X combination, the individual in-control ATS of the Shewhart X chart was taken to be β times the individual in-control ATS of the EWMA_X chart, where $\beta=1,5$, or 10. The reason for considering $\beta>1$ here is that in applications, the control limit parameter h_X of the Shewhart X chart is sometimes taken to be relatively large when this chart is used with an EWMA chart (see, e.g., Lucas 1982; Ryan 2000). Using a large value of h_X means that adding the Shewhart X chart will not have much effect on the false-alarm rate, and, when $n=1$, there will not be as much of a negative effect on the robustness of the combination to nonnormal distributions. Rather than use specific values of h_X for the numerical results given here, we specify relatively large in-control ATS values for the Shewhart X chart by specifying the value of β.

In setting up the comparisons of charts, we selected two EWMA_X charts with $n=4$ and $d=4.0$ as the basic EWMA reference charts, with one chart having $\lambda_\mu=.1$ and the other having $\lambda_\mu=.4$. The EWMA charts for monitoring σ used with the EWMA_X have either $\lambda_\sigma=.1$ or $\lambda_\sigma=.4$. Thus the four values of $(\lambda_\mu,\lambda_\sigma)$ that we used are $(.1,.1)$, $(.4,.4)$, $(.1,.4)$, and $(.4,.1)$.

To make comparisons of the case where $n=4$ and $d=4.0$ with the case where $n=1$ and $d=1.0$, (1) was used to determine that $\lambda=.02600$ for $n=1$ corresponds to $\lambda=.1$ for $n=4$ and that $\lambda=.11989$ for $n=1$ corresponds to $\lambda=.4$ when $n=4$. The value of $\lambda=.02600$ may seem to be relatively small compared with what is commonly used in applications, but it corresponds to $\lambda=.1$ when $n=4$. Moreover, it seems desirable to include relatively small values of λ, because the addition of Shewhart limits presumably will be of most help when λ is small.

For sustained shifts, we measured the expected time required for detection using the steady-state ATS (SSATS), which assumes that the control statistic has reached its steady-state distribution when the shift occurs and that the point of the shift within the time interval between two successive samples is distributed uniformly over the interval. Additional discussion on the SSATS was given by Reynolds and Stoumbos (2004a).

In most cases, simulation (with 1,000,000 runs) was used to evaluate the statistical properties of combinations of charts. The SSATS was simulated by generating 400 initial in-control observations before the shift was introduced, with any sequence that generated a false alarm in these 400 initial observations discarded.

In most applications, the control limits used for control charts are determined assuming a normal distribution for process observations, so a relevant performance measure is the robustness of control charts to deviations from normality. Robustness to nonnormality of the EWMA chart for monitoring μ was studied by Borror, Montgomery, and Runger (1999) and Stoumbos and Sullivan (2002), but these authors did not consider the more complex robustness problem of simultaneously monitoring μ and σ. Stoumbos and Reynolds (2000) investigated the robustness of various EWMA chart combinations for monitoring μ and σ, including EWMA combinations with Shewhart limits, but did not include several chart combinations that are of interest here. The most significant effect of nonnormal distributions on the properties of control charts is on the false-alarm rate, so we investigated the effect of various nonnormal process distributions on the in-control ATS.

In evaluating robustness, we considered the Laplace (double-exponential), t, gamma, and beta distributions. The t distribution with ν degrees of freedom is represented as $t(\nu)$, where we used $\nu = 4$ and 10. The gamma distribution with shape parameter α and scale parameter 1 is represented as Gam(α, 1), where we used $\alpha = 1$ (the exponential distribution), 2, and 4. The beta distribution with parameters α and β is represented as Beta(α, β), where we used $\alpha = \beta = 3$ and 4. Each of the nonnormal distributions was scaled to give an in-control standard deviation of σ_0, which was taken to be simply 1.0 in the simulations.

When there is a transient shift, the SSATS is not a particularly appropriate measure of control chart performance, because there is sometimes a substantial probability that the control chart will not signal while the transient shift is present. We measure control chart performance for transient shifts by the probability that a signal occurs during the duration l of the transient shift or shortly thereafter.

5. CHART COMPARISONS WHEN $n = 1$

The conclusions about adding Shewhart limits are the most straightforward when $n = 1$, so we start with this case. Table 1 gives SSATS values for various shifts in μ or σ for 10 charts or combinations of charts, where the EWMA charts have $(\lambda_\mu, \lambda_\sigma) = (.02600, .02600)$ (except for the column labeled 7, which we explain later). The matched in-control ATS values given first for the 10 charts are 1,481.6 (or very close to 1,481.6 due to very minor simulation error), corresponding to the case in which the distribution of the observations is normal. The bottom part of Table 1 gives in-control ATS values for eight nonnormal distributions.

First, consider comparisons based on the SSATS. In Table 1, comparing column 1 for the Shewhart X chart with column 2 for the EWMA$_X$ chart shows that, as expected, the EWMA$_X$ chart is much better than the Shewhart X chart for detecting small shifts in μ, but not as good for detecting large shifts. Columns 3–5 in Table 1 are for the combination of the Shewhart X and EWMA$_X$ charts with $\beta = 1, 5$, and 10. When $\beta = 1$, we see that the combination is almost as good as the EWMA$_X$ chart for detecting small shifts in μ, and almost as good as the Shewhart X chart for detecting large shifts in μ. Increasing the value of β improves the performance slightly for small shifts in μ, and hurts the performance slightly for large shifts. We conclude that the combination of the Shewhart X and EWMA$_X$ charts seems to have the advantages of both individual charts, providing good detection over the range of shifts in μ from small to large. Thus the widespread recommendation to add Shewhart limits to an EWMA chart seems to be justified if we consider only the problem of monitoring μ.

It is almost always desirable to monitor σ in addition to μ, so it is necessary to evaluate the effectiveness of control chart combinations for monitoring both μ and σ. From Table 1, we see that the combination of the Shewhart X and EWMA$_X$ charts will detect large increases in σ, but this combination is not very effective for detecting small increases in σ. For example, the expected time required to detect a 40% increase in σ is 77.3 hours when $\beta = 1$, and even longer when $\beta > 1$. Thus, for effective detection of small increases in σ, it is necessary to consider combinations of charts that have one chart designed specifically for detecting increases in σ. The best EWMA chart for detecting increases in σ is the EWMA$_{X^2}$ chart, so we consider the EWMA$_X$ and EWMA$_{X^2}$ chart combination.

Column 6 of Table 1 contains SSATS values for the combination of the EWMA$_X$ and EWMA$_{X^2}$ charts with $(\lambda_\mu, \lambda_\sigma) = (.02600, .02600)$. Comparing this column with column 3 for the EWMA$_X$ and Shewhart X combination with $\beta = 1$ shows that the EWMA$_X$ and EWMA$_{X^2}$ chart combination is a little better for detecting small and intermediate shifts in μ, but not quite as good for detecting shifts of size $\delta = 4.0$ and 5.0.

A Shewhart chart is an EWMA chart with $\lambda = 1$, so the addition of Shewhart limits to the EWMA chart can be thought of as a combination of two EWMA charts with $\lambda_\mu = .02600$ and $\lambda_\mu = 1$. The overall effect is that this combination is tuned to detect slightly larger shifts than the EWMA chart with $\lambda_\mu = .02600$ and smaller shifts than the Shewhart chart (the EWMA chart with $\lambda_\mu = 1$). Thus it may be necessary to increase $(\lambda_\mu, \lambda_\sigma)$ for the EWMA$_X$ and EWMA$_{X^2}$ chart combination to make this combination more comparable to the EWMA$_X$ and Shewhart X combination. Accordingly, $(\lambda_\mu, \lambda_\sigma)$ was increased to (.05723, .05132) for the EWMA$_X$ and EWMA$_{X^2}$ chart combination, and the SSATS values are given in column 7. It does not seem possible to get an extremely close match of the SSATS profiles for the two combinations of charts, but comparing the EWMA$_X$ and EWMA$_{X^2}$ chart combination in columns 6 and 7 with the EWMA$_X$ and Shewhart X chart combination in columns 3–5 demonstrates that the performances for detecting shifts in μ are roughly comparable. However, the EWMA$_X$ and EWMA$_{X^2}$ combination is much better for detecting small increases in σ. Thus we conclude that the EWMA$_X$ and EWMA$_{X^2}$ chart combination has better overall SSATS performance than the EWMA$_X$ and Shewhart X chart combination.

We have concluded that the EWMA$_X$ and EWMA$_{X^2}$ chart combination has very good performance, so the logical next question to consider is whether it is helpful to add the Shewhart X chart to this combination to make a combination of three control charts. SSATS values for this three-chart combination are given in columns 8–10 of Table 1. Comparing columns 8–10 with columns 6 and 7 shows that column 8 is a reasonably close match to column 7 (except possibly at the very small shift of $\delta = .25$), and that column 10 is a reasonably close match to column 6. Thus it appears that there is essentially no gain in overall SSATS performance when the Shewhart X chart is added to the EWMA$_X$ and EWMA$_{X^2}$ combination.

Now consider the effect on robustness of the addition of a Shewhart X chart to an EWMA chart or combination of charts. The in-control ATS values for the nonnormal distributions given at the bottom of Table 1 show that the Shewhart X chart is not robust, but the EWMA$_X$ chart is robust if λ is small. For example, the in-control ATS of the Shewhart X chart when the observations are $t(4)$ is only 116.3 hours, but the in-control ATS of the EWMA$_X$ chart is 1,265.4, a relatively small drop from the nominal value of 1,481.6.

When the Shewhart X chart is used with the EWMA$_X$ chart, the nonrobustness of the Shewhart X chart carries over to the combination, resulting in a very nonrobust combination. For example, the in-control ATS of the Shewhart X and EWMA$_X$ chart combination is only 137.6 hours for the $t(4)$ distribution

Table 1. In-Control ATS and Out-of-Control SSATS Values for Sustained Shifts in μ or σ for Matched Shewhart and EWMA Control Chart Combinations When $n = 1$, $d = 1.0$, and $(\lambda_\mu, \lambda_\sigma) = (.02600, .02600)$

	Size of shift		X chart	$EWMA_X$	X chart and $EWMA_X$			$EWMA_X$ and $EWMA_{X^2}$		X chart, $EWMA_X$, and $EWMA_{X^2}$		
				$\lambda_\mu = .02600$	$\lambda_\mu = .02600$			$\lambda_\mu = .02600$, $\lambda_\sigma = .02600$	$\lambda_\mu = .05723$, $\lambda_\sigma = .05132$	$\lambda_\mu = .02600$, $\lambda_\sigma = .02600$		
					$\beta = 1$	$\beta = 5$	$\beta = 10$			$\beta = 1$	$\beta = 5$	$\beta = 10$
Performance measure	$\delta = \frac{\lvert\mu-\mu_0\rvert}{\sigma_0}$	$\psi = \frac{\sigma}{\sigma_0}$	Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9	Column 10
Normal ATS	0	1.00	1,481.6	1,481.6	1,481.1	1,481.6	1,481.5	1,481.5	1,481.4	1,481.7	1,481.5	1,481.5
Normal SSATS	.25	1.00	1,053.3	108.6	132.4	114.2	111.5	131.3	186.7	145.3	133.6	132.2
	.50	1.00	521.3	36.8	41.4	37.9	37.4	41.0	43.2	43.5	41.4	41.2
	.75	1.00	246.6	21.4	23.5	21.9	21.7	23.1	21.0	24.2	23.3	23.1
	1.00	1.00	121.3	15.1	16.2	15.3	15.2	15.6	13.4	16.3	15.7	15.6
	1.50	1.00	34.3	9.4	9.6	9.3	9.3	8.6	7.3	8.9	8.6	8.6
	2.00	1.00	11.9	6.8	6.3	6.4	6.5	5.2	4.5	5.2	5.2	5.2
	3.00	1.00	2.4	4.4	2.5	3.0	3.2	2.3	2.1	2.1	2.2	2.3
	4.00	1.00	.9	3.2	1.0	1.2	1.4	1.3	1.1	1.0	1.1	1.2
	5.00	1.00	.6	2.6	.6	.6	.7	.8	.7	.6	.7	.7
	7.00	1.00	.5	1.8	.5	.5	.5	.5	.5	.5	.5	.5
	10.00	1.00	.5	1.3	.5	.5	.5	.5	.5	.5	.5	.5
	15.00	1.00	.5	.8	.5	.5	.5	.5	.5	.5	.5	.5
	0	1.20	216.3	459.8	246.9	320.4	355.1	90.7	108.2	97.3	91.7	91.1
	0	1.40	65.4	222.4	77.3	106.8	124.1	30.1	31.5	31.1	30.2	30.1
	0	1.60	29.3	133.8	34.6	47.4	55.4	16.9	16.5	17.0	16.8	16.8
	0	1.80	16.5	90.9	19.3	25.7	29.8	11.4	10.8	11.3	11.3	11.3
	0	2.00	10.7	66.7	12.4	16.1	18.3	8.5	8.0	8.2	8.4	8.4
	0	2.40	5.9	41.5	6.7	8.2	9.2	5.6	5.1	5.2	5.4	5.4
	0	3.00	3.4	24.8	3.7	4.4	4.8	3.6	3.3	3.3	3.5	3.5
	0	5.00	1.5	9.2	1.6	1.8	1.8	1.8	1.7	1.6	1.7	1.7
	0	7.00	1.1	5.2	1.1	1.2	1.3	1.3	1.2	1.1	1.2	1.2
	0	10.00	.9	3.0	.9	.9	1.0	1.0	.9	.9	.9	.9
	0	15.00	.7	1.8	.7	.8	.8	.8	.8	.7	.8	.8
Normal ATS	0	1.00	1,481.6	1,481.6	1,481.1	1,481.6	1,481.5	1,481.5	1,481.4	1,481.7	1,481.5	1,481.5
Laplace ATS	0	1.00	122.5	1,364.1	154.1	216.3	256.6	282.6	227.6	162.3	211.5	232.1
$t(4)$ ATS	0	1.00	116.3	1,265.4	137.6	175.0	197.2	251.4	205.2	146.4	184.4	201.2
$t(10)$ ATS	0	1.00	286.6	1,444.2	359.9	496.0	576.3	606.5	511.4	377.9	482.4	522.7
Gam(1, 1) ATS	0	1.00	81.4	1,330.9	96.9	126.4	144.9	192.4	153.7	103.6	133.7	147.6
Gam(2, 1) ATS	0	1.00	115.9	1,409.9	142.4	193.9	227.2	293.1	233.1	153.5	203.8	226.9
Gam(4, 1) ATS	0	1.00	174.9	1,442.8	219.9	307.7	365.1	453.9	363.5	238.4	322.2	359.4
Beta(3, 3) ATS	0	1.00	∞	1,514.2	3,021.0	1,809.4	1,658.0	2,798.0	3,169.4	4,048.6	2,990.6	2,876.5
Beta(4, 4) ATS	0	1.00	∞	1,509.4	3,001.0	1,804.7	1,653.4	2,632.4	2,967.3	3,825.0	2,811.1	2,706.3

when $\beta = 1$. This means that when the control limits of the charts are set up assuming a normal distribution for the process observations but the actual distribution is $t(4)$, then the false-alarm rate is $1{,}481.6/137.6 = 10.8$ times higher than expected. This is clearly unacceptable if it is desirable to have a chart combination that is robust to nonnormal distributions. Using a larger value of β helps slightly, but the combination is still not robust. The combination of the EWMA$_X$ and EWMA$_{X^2}$ is also not very robust, but this combination is at least as robust as the combinations that include the Shewhart X chart.

The basic cause of the lack of robustness of the charts in Table 1 is that charts designed to monitor σ for a normal distribution are not robust. When charts designed to monitor σ are used in combination with charts for monitoring μ, the resulting combinations are not robust. If robustness to nonnormal distributions is important, then the charts used to monitor σ must be modified. Stoumbos and Reynolds (2000) and Reynolds and Stoumbos (2004a) recommended using an EWMA chart for σ based on absolute deviations from target. Although absolute deviations from target are not quite as effective as squared deviations from target for detecting changes in σ when the distribution is normal, absolute deviations are much more robust to nonnormal distributions.

Table 2 gives SSATS results for the same charts as in Table 1, except that the charts in Table 2 are tuned to detect larger shifts, with $(\lambda_\mu, \lambda_\sigma) = (.11989, .11989)$ [column 7 has $(\lambda_\mu, \lambda_\sigma) = (.13110, .13110)$]. Looking at columns 2 and 3 in Tables 1 and 2 shows that adding the Shewhart X chart to the EWMA$_X$ chart is of less help when λ is larger. For example, in Table 1 the EWMA$_X$ chart has an SSATS of 2.6 when $\delta = 5.0$, and adding the Shewhart X chart reduces the SSATS to .6. In Table 2 the EWMA$_X$ chart has an SSATS of 1.4 when $\delta = 5.0$, so adding the Shewhart X chart and reducing the SSATS to .6 is less impressive in this case.

Comparing different combinations of charts in Table 2, we obtain conclusions similar to those obtained from Table 1. In Table 2 the EWMA$_X$ and EWMA$_{X^2}$ chart combination in columns 6 and 7 has roughly the same performance for detecting shifts in μ as the EWMA$_X$ and Shewhart X chart combination in columns 3–5, but the EWMA$_X$ and EWMA$_{X^2}$ combination is much better for detecting small increases in σ. Thus the EWMA$_X$ and EWMA$_{X^2}$ combination seems preferable to the EWMA$_X$ and Shewhart X combination. Comparing columns 8–10 with columns 6 and 7 shows that there is little to be gained in terms of SSATS from adding the Shewhart X chart to the EWMA$_X$ and EWMA$_{X^2}$ chart combination. In fact, the SSATS values in column 7 of Table 2 are lower or as low as those in column 8 for all shifts in μ or σ, except for the one case of $\delta = 4.0$.

Comparing the in-control ATS values for nonnormal distributions in column 2 of Tables 1 and 2, we see that the effect of increasing the value of λ_μ from .02600 to .11989 is a considerable reduction in the robustness of the EWMA$_X$ chart. Comparing the ATS values for the combinations of charts in Tables 1 and 2 shows some decrease in robustness as λ_μ and λ_σ increase, but the decrease is not very substantial because these combinations are not very robust, even when very small values of λ_μ and λ_σ are used.

Table 3 gives SSATS values for combinations of charts for μ and σ, where one EWMA in the combination is tuned to detect small shifts and the other EWMA chart is tuned to detect large shifts. In particular, the charts in columns 1, 3, 4, and 5 have $(\lambda_\mu, \lambda_\sigma) = (.02600, .11989)$ [column 2 has $(\lambda_\mu, \lambda_\sigma) = (.04840, .13494)$], and the charts in columns 6 and 8–10 have $(\lambda_\mu, \lambda_\sigma) = (.11989, .02600)$ [column 2 has $(\lambda_\mu, \lambda_\sigma) = (.13110, .04265)$]. Note that Table 3 does not include the X chart, the EWMA$_X$ chart, or the EWMA$_X$ chart and Shewhart X chart combination, because the relevant ATS and SSATS results for these charts can be obtained from Table 1 or 2.

From columns 1 and 2 of Table 3, we see that tuning the EWMA$_X$ chart for small shifts and the EWMA$_{X^2}$ chart for larger shifts gives a combination that is very effective for detecting a wide range of shifts in μ. The EWMA$_X$ chart effectively detects small shifts in μ, and the EWMA$_{X^2}$ chart effectively detects large shifts in μ. Comparing the EWMA$_X$ and EWMA$_{X^2}$ chart combination in column 1 of Table 3 with the corresponding EWMA$_X$ and Shewhart X chart combination in column 3 of Table 1 shows that the EWMA$_X$ and EWMA$_{X^2}$ chart combination has better (or the same) SSATS performance for all shifts in μ and σ, and much better performance for small shifts in σ.

In Table 3, comparing columns 6 and 7 with columns 1 and 2 shows that tuning the EWMA$_X$ chart for large shifts and the EWMA$_{X^2}$ chart for small shifts gives a combination that has good performance for intermediate shifts in μ and small shifts in σ, but performance is not as good for small or large shifts in μ and large shifts in σ.

In general, the SSATS results for the EWMA$_X$ and EWMA$_{X^2}$ chart combination in Tables 1–3 show that the choice of $(\lambda_\mu, \lambda_\sigma)$ provides considerable flexibility for tuning the combination to detect any particular pattern of shifts in μ and σ that may be of interest. The EWMA$_X$ and EWMA$_{X^2}$ chart combination was set up here with each chart having the same individual in-control ATS, so another option for providing flexibility is to allow the two charts to have different individual in-control ATS values while maintaining the required in-control ATS value for the combination. We do not discuss this second option here.

Our conclusions for the case of $n = 1$ can be summarized as follows. Adding Shewhart limits to the EWMA$_X$ chart improves overall performance, but even better overall performance is achieved by using the combination of the EWMA$_X$ and EWMA$_{X^2}$ charts. Adding the Shewhart X chart to the EWMA$_X$ and EWMA$_{X^2}$ charts to make a three-chart combination does not seem justified based on statistical performance. The three-chart combination is more complex and less robust than the EWMA$_X$ and EWMA$_{X^2}$ combination, and there is no significant gain in the ability to detect shifts in μ or σ.

Many practitioners are very familiar and comfortable with Shewhart charts, and may prefer to continue using the X chart for reasons that do not directly show up in performance measures such as the SSATS. Our results show that if the Shewhart X chart is to be used with the EWMA$_X$ chart, then the EWMA$_{X^2}$ chart should also be used. All of the combinations with the best overall performance in Tables 1–3 included the EWMA$_{X^2}$ chart. Thus if the objective is to detect both small and large changes in μ or σ, then the important chart for improving the performance of the EWMA$_X$ chart is the EWMA$_{X^2}$ chart, and the addition of the Shewhart X chart should be considered optional, depending on the user's preference.

Table 2. In-Control ATS and Out-of-Control SSATS Values for Sustained Shifts in μ or σ for Matched Shewhart and EWMA Control Chart Combinations When $n = 1$, $d = 1.0$, and $(\lambda_\mu, \lambda_\sigma) = (.11989, .11989)$

	Size of shift		X chart	$EWMA_X$	X chart and $EWMA_X$			$EWMA_X$ and $EWMA_{X^2}$		X chart, $EWMA_X$, and $EWMA_{X^2}$		
				$\lambda_\mu = .11989$	$\lambda_\mu = .11989$			$\lambda_\mu = .11989$, $\lambda_\sigma = .11989$	$\lambda_\mu = .13110$, $\lambda_\sigma = .13110$	$\lambda_\mu = .11989$, $\lambda_\sigma = .11989$		
					$\beta = 1$	$\beta = 5$	$\beta = 10$			$\beta = 1$	$\beta = 5$	$\beta = 10$
Performance measure	$\delta = \frac{\lvert\mu-\mu_0\rvert}{\sigma_0}$	$\psi = \frac{\sigma}{\sigma_0}$	Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9	Column 10
Normal ATS	0	1.00	1,481.6	1,481.6	1,481.4	1,481.7	1,481.7	1,481.2	1,481.3	1,481.6	1,481.1	1,481.2
Normal SSATS	.25	1.00	1,053.3	227.7	312.3	247.0	237.2	309.1	329.7	347.6	310.2	309.1
	.50	1.00	521.3	48.4	61.3	51.4	49.9	60.7	64.7	66.6	60.9	60.7
	.75	1.00	246.6	20.2	23.4	20.9	20.5	23.2	24.0	24.7	23.3	23.2
	1.00	1.00	121.3	11.8	13.1	12.1	12.0	13.0	13.1	13.6	13.0	13.0
	1.50	1.00	34.3	6.3	6.6	6.3	6.3	6.4	6.4	6.6	6.4	6.4
	2.00	1.00	11.9	4.2	4.2	4.1	4.2	4.0	3.9	4.0	4.0	4.0
	3.00	1.00	2.4	2.5	2.0	2.1	2.2	1.8	1.8	1.8	1.8	1.8
	4.00	1.00	.9	1.8	.9	1.1	1.2	1.0	1.0	.9	1.0	1.0
	5.00	1.00	.6	1.4	.6	.6	.7	.6	.6	.6	.6	.6
	7.00	1.00	.5	1.0	.5	.5	.5	.5	.5	.5	.5	.5
	10.00	1.00	.5	.6	.5	.5	.5	.5	.5	.5	.5	.5
	15.00	1.00	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5
	0	1.20	216.3	310.2	217.3	252.1	267.5	142.3	146.0	150.7	142.7	142.3
	0	1.40	65.4	120.3	67.0	81.6	89.2	38.3	39.3	40.3	38.4	38.3
	0	1.60	29.3	63.6	30.4	37.3	41.2	18.0	18.3	18.6	18.1	18.0
	0	1.80	16.5	40.0	17.3	21.1	23.2	11.1	11.2	11.3	11.1	11.1
	0	2.00	10.7	28.0	11.3	13.6	14.9	7.9	7.9	7.9	7.9	7.9
	0	2.40	5.9	16.5	6.2	7.3	7.9	4.9	4.9	4.9	4.9	4.9
	0	3.00	3.4	9.6	3.6	4.1	4.4	3.1	3.1	3.1	3.1	3.1
	0	5.00	1.5	3.6	1.6	1.7	1.8	1.6	1.5	1.5	1.6	1.6
	0	7.00	1.1	2.2	1.1	1.2	1.2	1.1	1.1	1.1	1.1	1.1
	0	10.00	.9	1.5	.9	.9	.9	.9	.9	.9	.9	.9
	0	15.00	.7	1.0	.7	.8	.8	.7	.7	.7	.7	.7
Normal ATS	0	1.00	1,481.6	1,481.6	1,481.4	1,481.7	1,481.7	1,481.2	1,481.3	1,481.6	1,481.1	1,481.2
Laplace ATS	0	1.00	122.5	709.2	150.4	204.8	238.6	182.4	178.7	148.9	179.2	182.4
$t(4)$ ATS	0	1.00	116.3	553.3	135.9	170.6	190.6	166.6	163.5	138.1	163.7	166.6
$t(10)$ ATS	0	1.00	286.6	1,071.9	351.0	471.0	538.5	421.8	413.8	349.8	416.0	421.8
Gam(1, 1) ATS	0	1.00	81.4	475.1	95.4	122.0	138.0	121.3	118.8	97.5	118.6	121.3
Gam(2, 1) ATS	0	1.00	115.9	657.0	139.1	184.8	212.7	180.5	176.2	142.2	176.7	180.5
Gam(4, 1) ATS	0	1.00	174.9	876.3	214.1	292.3	339.5	280.6	273.5	220.0	275.5	280.6
Beta(3, 3) ATS	0	1.00	∞	2,031.9	4,421.2	2,467.2	2,237.0	4,252.0	4,506.7	5,812.0	4,292.4	4,252.0
Beta(4, 4)	0	1.00	∞	1,902.2	4,041.5	2,293.3	2,088.8	3,837.9	4,018.7	5,212.0	3,874.2	3,837.9
Control chart parameters	h_X		3.3996		3.5770	3.8582	4.0047			3.6373	3.9728	4.1329
	h_{EX}			3.2237	3.4265	3.2760	3.2500	3.4175	3.4311	3.4943	3.4199	3.4175
	h_{EX^2}							5.1129	5.2298	5.3014	5.1187	5.1129

Table 3. In-Control ATS and Out-of-Control SSATS Values for Sustained Shifts in μ or σ for Matched Shewhart and EWMA Control Chart Combinations When $n = 1$, $d = 1.0$, and $(\lambda_\mu, \lambda_\sigma) = (.02600, .11989), (.11989, .02600)$

			EWMA$_X$ and EWMA$_{X^2}$		X chart, EWMA$_X$, and EWMA$_{X^2}$			EWMA$_X$ and EWMA$_{X^2}$		X chart, EWMA$_X$, and EWMA$_{X^2}$		
	Size of shift		$\lambda_\mu = .02600$, $\lambda_\sigma = .11989$	$\lambda_\mu = .04840$, $\lambda_\sigma = .13494$	$\lambda_\mu = .02600$, $\lambda_\mu = .11989$			$\lambda_\mu = .11989$, $\lambda_\sigma = .02600$	$\lambda_\mu = .13110$, $\lambda_\sigma = .04265$	$\lambda_\mu = .11989$, $\lambda_\sigma = .02600$		
					$\beta = 1$	$\beta = 5$	$\beta = 10$			$\beta = 1$	$\beta = 5$	$\beta = 10$
Performance measure	$\delta = \frac{\lvert\mu-\mu_0\rvert}{\sigma_0}$	$\psi = \frac{\sigma}{\sigma_0}$	Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9	Column 10
Normal ATS	0	1.00	1,481.3	1,481.2	1,481.3	1,481.6	1,481.3	1,481.1	1,481.8	1,481.6	1,481.4	1,481.4
Normal SSATS	.25	1.00	132.1	171.0	142.3	132.4	132.1	302.0	324.5	353.7	310.6	305.4
	.50	1.00	41.2	42.1	43.0	41.2	41.2	59.9	64.1	67.9	61.2	60.5
	.75	1.00	23.1	21.3	24.0	23.2	23.1	23.1	23.9	25.1	23.4	23.3
	1.00	1.00	15.6	13.8	16.1	15.6	15.6	13.0	13.1	13.8	13.1	13.0
	1.50	1.00	8.3	7.5	8.6	8.3	8.3	6.5	6.4	6.8	6.6	6.6
	2.00	1.00	4.7	4.4	4.8	4.7	4.7	4.2	4.0	4.2	4.2	4.2
	3.00	1.00	1.9	1.8	1.8	1.9	1.9	2.1	2.0	1.9	2.0	2.1
	4.00	1.00	1.0	1.0	.9	1.0	1.0	1.3	1.1	1.0	1.1	1.1
	5.00	1.00	.6	.6	.6	.6	.6	.8	.7	.6	.6	.7
	7.00	1.00	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5
	10.00	1.00	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5
	15.00	1.00	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5
	0	1.20	150.4	153.1	158.3	150.7	150.4	88.7	100.5	95.5	89.8	89.2
	0	1.40	39.9	41.0	41.9	40.0	39.9	29.4	30.3	30.6	29.6	29.5
	0	1.60	18.6	18.9	19.2	18.6	18.6	16.5	16.2	16.6	16.4	16.4
	0	1.80	11.4	11.5	11.6	11.4	11.4	11.2	10.7	11.0	11.0	11.1
	0	2.00	8.0	8.1	8.1	8.0	8.0	8.3	7.9	8.1	8.2	8.2
	0	2.40	5.0	5.0	4.9	5.0	5.0	5.4	5.1	5.1	5.3	5.3
	0	3.00	3.2	3.2	3.1	3.2	3.2	3.6	3.4	3.3	3.4	3.4
	0	5.00	1.6	1.6	1.5	1.6	1.6	1.7	1.7	1.6	1.6	1.7
	0	7.00	1.1	1.1	1.1	1.1	1.1	1.3	1.2	1.1	1.2	1.2
	0	10.00	.9	.9	.9	.9	.9	1.0	.9	.9	.9	.9
	0	15.00	.7	.7	.7	.7	.7	.8	.8	.7	.8	.8
Normal ATS	0	1.00	1,481.3	1,481.2	1,481.3	1,481.6	1,481.3	1,481.1	1,481.8	1,481.6	1,481.4	1,481.4
Laplace ATS	0	1.00	186.9	181.7	151.7	183.6	186.9	270.0	232.9	158.8	204.7	223.9
t(4) ATS	0	1.00	169.2	164.9	139.6	166.2	169.2	243.4	211.3	144.5	180.6	196.6
t(10) ATS	0	1.00	433.2	422.0	357.1	427.0	433.2	580.4	517.2	369.4	466.7	503.9
Gam(1, 1) ATS	0	1.00	123.0	119.8	98.6	120.2	123.0	187.4	159.5	102.2	131.1	144.4
Gam(2, 1) ATS	0	1.00	183.7	178.7	144.5	179.7	183.7	282.2	240.0	150.5	198.4	219.9
Gam(4, 1) ATS	0	1.00	286.4	277.9	223.9	280.4	286.4	434.4	372.2	232.9	312.5	347.2
Beta(3, 3) ATS	0	1.00	2,989.7	3,184.4	3,895.2	3,014.5	2,989.7	3,898.7	4,276.4	6,006.0	4,202.2	4,021.9
Beta(4, 4) ATS	0	1.00	2,937.9	3,088.4	3,831.9	2,962.7	2,937.9	3,347.4	3,669.7	5,111.3	3,606.7	3,453.5
Control chart parameters	h_X				3.6443	3.9808	4.1408			3.6570	3.9825	4.1360
	h_{EX}		3.0780	3.2402	3.1661	3.0808	3.0780	3.4140	3.4280	3.5164	3.4321	3.4215
	h_{EX^2}		5.1376	5.2923	5.3210	5.1434	5.1376	3.6176	4.0065	3.8034	3.6505	3.6312

6. CHART COMPARISONS WHEN $n = 4$

In the case where $n = 1$, it was possible to get reasonably close matches of SSATS profiles for the purpose of comparing combinations of control charts, but obtaining close matches turned out to be more difficult when $n = 4$. Thus when $n = 4$, it is sometimes more difficult to say that one chart combination is better than another.

Table 4 gives ATS and SSATS values for 10 charts or combinations of charts involving EWMA and Shewhart charts when $n = 4$ and $(\lambda_\mu, \lambda_\sigma) = (.10, .10)$. This value of $(\lambda_\mu, \lambda_\sigma)$ for $n = 4$ corresponds to the value (.02600, .02600) used in Table 1 for $n = 1$. The basic structure of Table 4 is the same as that of Table 1, except that some of the combinations of charts considered for $n = 4$ are different than those considered for $n = 1$. As in Table 1, one of the EWMA_X and EWMA_{X^2} chart combinations in Table 4 has an increased value of $(\lambda_\mu, \lambda_\sigma)$ to more closely match the SSATS profiles of the combinations with the Shewhart limit. Table 5 has the same charts as Table 4, except that $(\lambda_\mu, \lambda_\sigma) = (.4, .4)$, corresponding to values of $(\lambda_\mu, \lambda_\sigma)$ for $n = 1$ in Table 2. Similarly, Table 6 has values of $(\lambda_\mu, \lambda_\sigma)$ corresponding to those in Table 3.

Column 1 of Tables 4 and 5 gives SSATS values for the EWMA_X chart, and columns 2 and 3 give SSATS values for the combination of the Shewhart $\bar{X}$ and EWMA_X charts with $\beta = 1$ and 5. The general effect of adding Shewhart limits to the EWMA_X chart when $n = 4$ is the same as that when $n = 1$, but there are differences in the size of the shift for which the effect is most pronounced. When $n = 1$, adding Shewhart limits gave a substantial reduction in SSATS for all shifts in μ of size $\delta \geq 3.0$. In Table 4 we see that with $n = 4$, the Shewhart limits are most helpful for $1.5 \leq \delta \leq 4.0$, with all of the combinations having SSATS values close to the minimum value of 2.0 when $\delta \geq 5.0$. In Table 5, where λ_μ is large, the effect of adding the Shewhart limits is relatively small.

When $n = 1$, the Shewhart X chart is effective for detecting large increases in σ, but when $n = 4$, the Shewhart $\bar{X}$ chart cannot be relied on to detect any increase in σ. Thus, when $n = 4$, it is necessary to consider chart combinations with one chart designed explicitly to detect increases in σ. SSATS values are given in column 4 of Tables 4 and 5 for the EWMA_X and EWMA_{S^2} chart combination, and in columns 5 and 6 for the $\text{EWMA}_X, \text{EWMA}_{S^2}$, and Shewhart $\bar{X}$ chart combination with $\beta = 1$ and 5. Adding Shewhart limits improves performance for intermediate shifts in μ and hurts performance slightly for small shifts in μ. Performance is improved slightly for most shifts in σ. We conclude that overall performance is improved by adding Shewhart limits to this combination. However, adding Shewhart limits gives a three-chart combination, and we argue next that this is not the best option.

Columns 7 and 8 in Tables 4 and 5 give SSATS values for the EWMA_X and EWMA_{X^2} chart combinations. Comparing column 8 with column 5 shows that the EWMA_X and EWMA_{X^2} chart combination generally has better performance for shifts in μ or σ than the three-chart combination of the Shewhart $\bar{X}$, EWMA_X, and EWMA_{S^2} charts, except for $\delta = .25$ and δ around 2.0. Even with the increased values of $(\lambda_\mu, \lambda_\sigma)$, the EWMA_X and EWMA_{X^2} chart combination does not perform quite as well as the Shewhart $\bar{X}$, EWMA_X, and EWMA_{S^2} chart combination around $\delta = 2.0$. Unless shifts of this size are of particular concern, we conclude that the better overall performance of the EWMA_X and EWMA_{X^2} two-chart combination for other shifts in μ or σ is sufficient to recommend the simpler two-chart combination over the more complicated three-chart combination for most applications.

It is worth commenting explicitly on the fact that the EWMA_X and EWMA_{X^2} chart combination is uniformly better than (or the same as) the EWMA_X and EWMA_{S^2} chart combination for detecting shifts in either μ or σ (compare columns 4 and 7 in Tables 4 and 5). Most work that considers EWMA charts for monitoring σ when $n > 1$ uses S_k^2 or some function of S_k^2, such as $\ln S_k^2$. Our results here show that it is better to use squared deviations from target instead of S_k^2. Using squared deviations from target gives an extra degree of freedom that significantly improves the ability to detect increases in σ when n is small. In addition, using squared deviations from target greatly improves the ability to detect large changes in μ.

The last question to consider is whether adding Shewhart limits to the EWMA_X and EWMA_{X^2} chart combination will give a significant improvement in performance. Columns 9 and 10 in Tables 4 and 5 and columns 5, 6, 11, and 12 in Table 6 give SSATS values for the combination consisting of the Shewhart $\bar{X}$ chart added to the EWMA_X and EWMA_{X^2} chart combination with $\beta = 1$ and 5. Adding the Shewhart limits hurts performance slightly for small shifts in μ, improves performance for intermediate shifts in μ, and hurts performance slightly for small shifts in σ. The choice between the EWMA_X and EWMA_{X^2} chart combination and the $\text{EWMA}_X, \text{EWMA}_{X^2}$, and Shewhart $\bar{X}$ chart combination is not clear-cut. However, our judgment is that the more complicated three-chart combination does not offer sufficient improvement in overall statistical performance over the two-chart combination to justify using the three-chart combination. As in the case of $n = 1$, if a practitioner desires to use a Shewhart chart because of familiarity, then the $\text{EWMA}_X, \text{EWMA}_{X^2}$, and Shewhart $\bar{X}$ chart combination will provide very good overall performance.

Note that if the Shewhart $\bar{X}$ chart is used with the EWMA_X chart, it is important to also use the EWMA_{X^2} chart instead of the EWMA_{S^2} chart. From Tables 4–6, we see that the $\text{EWMA}_X, \text{EWMA}_{X^2}$, and Shewhart $\bar{X}$ chart combination has uniformly better (or the same) performance than the $\text{EWMA}_X, \text{EWMA}_{S^2}$, and Shewhart $\bar{X}$ chart combination.

Many practitioners assume that using a sample of size $n = 4$ will give a reasonable level of robustness in control charts. Examining the in-control ATS values for the nonnormal distributions at the bottom of Tables 4–6 shows that the only chart that can be considered robust is the EWMA_X chart used alone. The EWMA_X and Shewhart $\bar{X}$ chart combination is moderately robust, but this combination does not satisfy the requirement for monitoring both μ and σ. We have recommended the EWMA_X and EWMA_{X^2} chart combination based on SSATS performance for normal distributions. Although this combination is not robust, its level of robustness is roughly comparable to the other combinations that could be used to monitor both μ and σ. If robust combinations for monitoring μ and σ are desired, then we recommend replacing the chart based on squared deviations with a chart based on absolute deviations from target, as discussed earlier.

Table 4. In-Control ATS and Out-of-Control SSATS Values for Sustained Shifts in μ or σ for Matched Shewhart and EWMA Control Chart Combinations When $n = 4$, $d = 4.0$, and $(\lambda_\mu, \lambda_\sigma) = (.10, .10)$

	Size of shift		$EWMA_X$	$\bar{X}$ chart and $EWMA_X$		$EWMA_X$ and $EWMA_{S^2}$	$\bar{X}$ chart, $EWMA_X$, and $EWMA_{S^2}$		$EWMA_X$ and $EWMA_{X^2}$		$\bar{X}$ chart, $EWMA_X$, and $EWMA_{X^2}$	
			$\lambda_\mu = .10$	$\lambda_\mu = .10$		$\lambda_\mu = .10, \lambda_\sigma = .10$	$\lambda_\mu = .10, \lambda_\sigma = .10$		$\lambda_\mu = .10$, $\lambda_\sigma = .10$	$\lambda_\mu = .21$, $\lambda_\sigma = .19$	$\lambda_\mu = .10, \lambda_\sigma = .10$	
				$\beta = 1$	$\beta = 5$		$\beta = 1$	$\beta = 5$			$\beta = 1$	$\beta = 5$
Performance measure	$\delta = \frac{\lvert\mu - \mu_0\rvert}{\sigma_0}$	$\psi = \frac{\sigma}{\sigma_0}$	Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9	Column 10
Normal ATS	0	1.00	1,481.6	1,482.1	1,481.3	1,481.6	1,481.4	1,481.6	1,481.9	1,481.8	1,481.6	1,481.6
Normal SSATS	.25	1.00	108.1	129.7	112.8	132.8	147.2	135.7	130.7	187.1	144.1	133.2
	.50	1.00	36.1	39.6	36.8	41.2	43.0	41.5	40.4	42.7	42.2	40.7
	.75	1.00	20.8	21.4	20.7	23.3	23.1	23.0	22.5	20.4	22.6	22.4
	1.00	1.00	14.5	13.6	13.8	16.1	14.8	15.3	15.1	12.8	14.3	14.7
	1.50	1.00	9.0	6.0	6.8	9.9	6.6	7.7	8.2	6.8	6.3	7.2
	2.00	1.00	6.5	3.0	3.5	7.1	3.2	3.9	4.9	4.2	3.1	3.7
	3.00	1.00	4.2	2.0	2.0	4.6	2.0	2.0	2.3	2.1	2.0	2.0
	4.00	1.00	2.8	2.0	2.0	3.2	2.0	2.0	2.0	2.0	2.0	2.0
	5.00	1.00	2.2	2.0	2.0	2.3	2.0	2.0	2.0	2.0	2.0	2.0
	7.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	10.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	15.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	0	1.20	483.9	334.5	392.4	109.7	112.4	108.4	89.8	107.0	97.4	91.0
	0	1.40	241.1	131.5	162.6	36.4	35.6	35.3	29.9	31.2	31.4	30.1
	0	1.60	147.8	69.2	86.2	20.1	19.1	19.1	16.7	16.3	17.3	16.8
	0	1.80	102.1	43.4	53.5	13.6	12.5	12.6	11.3	10.7	11.6	11.3
	0	2.00	76.0	30.4	37.0	10.1	9.1	9.2	8.5	7.9	8.6	8.4
	0	2.40	48.4	18.3	21.7	6.6	5.8	5.9	5.6	5.1	5.6	5.5
	0	3.00	30.1	11.4	13.0	4.4	3.8	3.9	3.8	3.5	3.8	3.7
	0	5.00	12.4	5.6	6.0	2.5	2.3	2.3	2.3	2.3	2.3	2.3
	0	7.00	7.8	4.1	4.4	2.2	2.1	2.1	2.1	2.1	2.1	2.1
	0	10.00	5.3	3.3	3.5	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	0	15.00	3.9	2.8	2.9	2.0	2.0	2.0	2.0	2.0	2.0	2.0
Normal ATS	0	1.00	1,481.6	1,482.1	1,481.3	1,481.6	1,481.4	1,481.6	1,481.9	1,481.8	1,481.6	1,481.6
Laplace ATS	0	1.00	1,357.2	704.2	898.7	380.0	356.5	361.3	302.6	252.0	314.0	303.6
$t(4)$ ATS	0	1.00	1,263.6	530.8	663.5	320.9	302.8	307.4	266.2	223.3	273.0	267.0
$t(10)$ ATS	0	1.00	1,432.2	1,056.2	1,205.2	736.9	701.3	710.0	629.9	544.7	646.9	630.9
Gam(1, 1) ATS	0	1.00	1,360.5	469.0	619.6	260.0	236.9	240.9	205.5	169.8	210.2	205.1
Gam(2, 1) ATS	0	1.00	1,424.0	639.5	819.5	397.1	353.0	363.3	308.7	253.7	312.0	306.1
Gam(4, 1) ATS	0	1.00	1,451.8	841.9	1,015.9	600.7	531.6	549.8	472.7	391.5	473.7	466.3
Beta(3, 3) ATS	0	1.00	1,517.4	2,045.1	1,694.4	2,633.0	3,015.6	2,807.2	2,771.5	3,071.3	2,995.1	2,895.2
Beta(4, 4) ATS	0	1.00	1,511.3	1,916.3	1,657.4	2,428.5	2,664.0	2,554.8	2,598.1	2,871.0	2,690.3	2,679.1
Control chart parameters	h_X			3.1903	3.5013		3.3103	3.6615			3.2953	3.6501
	h_{S^2}											
	h_{EX}		2.7015	2.9379	2.7598	2.9518	3.0823	2.9813	2.9415	3.0855	3.0643	2.9657
	h_{ES^2}					3.5122	3.7220	3.5593				
	h_{EX^2}								3.4142	3.8264	3.6003	3.4502

Table 5. In-Control ATS and Out-of-Control SSATS Values for Sustained Shifts in μ or σ for Matched Shewhart and EWMA Control Chart Combinations When $n = 4$, $d = 4.0$, and $(\lambda_\mu, \lambda_\sigma) = (.40, .40)$

Performance measure	Size of shift: $\delta = \frac{\lvert\mu-\mu_0\rvert}{\sigma_0}$	Size of shift: $\psi = \frac{\sigma}{\sigma_0}$	$EWMA_X$ $\lambda_\mu = .40$	$\bar{X}$ chart and $EWMA_X$ $\lambda_\mu = .40$, $\beta = 1$	$\bar{X}$ chart and $EWMA_X$ $\lambda_\mu = .40$, $\beta = 5$	$EWMA_X$ and $EWMA_{S^2}$ $\lambda_\mu = .40, \lambda_\sigma = .40$	$\bar{X}$ chart, $EWMA_X$, and $EWMA_{S^2}$ $\lambda_\mu = .40, \lambda_\sigma = .40$, $\beta = 1$	$\bar{X}$ chart, $EWMA_X$, and $EWMA_{S^2}$ $\lambda_\mu = .40, \lambda_\sigma = .40$, $\beta = 5$	$EWMA_X$ and $EWMA_{X^2}$ $\lambda_\mu = .40$, $\lambda_\sigma = .40$	$EWMA_X$ and $EWMA_{X^2}$ $\lambda_\mu = .43$, $\lambda_\sigma = .43$	$\bar{X}$ chart, $EWMA_X$, and $EWMA_{X^2}$ $\lambda_\mu = .40, \lambda_\sigma = .40$, $\beta = 1$	$\bar{X}$ chart, $EWMA_X$, and $EWMA_{X^2}$ $\lambda_\mu = .40, \lambda_\sigma = .40$, $\beta = 5$
			Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9	Column 10
Normal ATS	0	1.00	1,481.6	1,481.4	1,481.3	1,481.3	1,481.5	1,481.4	1,481.4	1,481.6	1,481.4	1,481.6
Normal SSATS	.25	1.00	230.0	294.1	240.7	326.5	370.3	334.0	311.9	333.0	351.5	317.1
	.50	1.00	48.2	57.8	49.7	62.6	69.4	63.7	60.6	64.7	66.8	61.4
	.75	1.00	19.5	21.7	19.8	23.3	24.5	23.4	22.7	23.5	23.9	22.8
	1.00	1.00	11.2	11.6	11.1	12.7	12.8	12.7	12.4	12.5	12.5	12.3
	1.50	1.00	5.7	5.0	5.3	6.3	5.5	5.9	5.9	5.8	5.4	5.7
	2.00	1.00	3.7	2.8	3.1	4.1	3.0	3.3	3.5	3.5	2.9	3.2
	3.00	1.00	2.1	2.0	2.0	2.2	2.0	2.0	2.1	2.0	2.0	2.0
	4.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	5.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	7.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	10.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	15.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	0	1.20	356.5	306.8	330.4	160.8	164.2	158.9	141.5	145.4	152.4	142.7
	0	1.40	148.1	117.4	129.9	44.7	44.7	43.7	37.7	38.7	40.2	38.0
	0	1.60	81.5	61.6	68.6	20.8	20.5	20.3	17.7	18.0	18.5	17.8
	0	1.80	52.7	38.8	43.3	12.6	12.3	12.2	10.9	11.0	11.3	10.9
	0	2.00	37.6	27.4	30.4	8.9	8.5	8.5	7.7	7.7	7.9	7.7
	0	2.40	23.0	16.8	18.5	5.5	5.3	5.3	4.9	4.9	5.0	4.9
	0	3.00	14.2	10.7	11.5	3.6	3.5	3.5	3.3	3.3	3.4	3.3
	0	5.00	6.5	5.4	5.7	2.3	2.2	2.2	2.2	2.2	2.2	2.2
	0	7.00	4.7	4.0	4.2	2.1	2.1	2.1	2.1	2.1	2.1	2.1
	0	10.00	3.6	3.3	3.4	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	0	15.00	3.0	2.8	2.8	2.0	2.0	2.0	2.0	2.0	2.0	2.0
Normal ATS	0	1.00	1,481.6	1,481.4	1,481.3	1,481.3	1,481.5	1,481.4	1,481.4	1,481.6	1,481.4	1,481.6
Laplace ATS	0	1.00	865.7	646.1	757.3	270.0	272.3	267.3	215.1	212.6	230.1	217.1
$t(4)$ ATS	0	1.00	680.7	496.3	579.4	230.3	231.9	228.7	190.8	188.6	200.4	192.1
$t(10)$ ATS	0	1.00	1,181.4	1,005.5	1,101.9	564.5	568.5	559.6	475.5	470.4	504.4	479.5
Gam(1, 1) ATS	0	1.00	648.6	427.1	519.2	180.7	179.7	177.4	142.5	140.7	149.8	143.3
Gam(2, 1) ATS	0	1.00	861.5	591.0	712.1	275.6	270.9	269.4	210.9	207.9	222.2	212.0
Gam(4, 1) ATS	0	1.00	1,074.6	797.3	929.3	430.3	419.7	418.7	325.1	320.5	343.0	236.8
Beta(3, 3) ATS	0	1.00	1,811.6	2,250.1	1,933.2	3,767.9	3,861.3	3,891.3	3,661.9	3,769.4	3,598.2	3,709.6
Beta(4, 4) ATS	0	1.00	1,738.1	2,053.1	1,829.5	3,362.1	3,329.9	3,435.8	3,362.3	3,439.4	3,169.5	3,368.4
Control chart parameters	h_X			3.1586	3.4844		3.2930	3.6551			3.2728	3.6380
	h_{S^2}											
	h_{EX}		2.9589	3.1257	2.9879	3.1737	3.2659	3.1891	3.1582	3.1630	3.2449	3.1691
	h_{ES^2}					4.6199	4.8456	4.6573				
	h_{EX^2}								4.3804	4.4357	4.5737	4.4045

Table 6. In-Control ATS and Out-of-Control SSATS Values for Sustained Shifts in μ or σ for Matched Shewhart and EWMA Control Chart Combinations When $n = 4$, $d = 4.0$, and $(\lambda_\mu, \lambda_\sigma) = (.10, .40), (.40, .10)$

			$\bar{X}$ chart, $EWMA_X$, and $EWMA_{S^2}$		$EWMA_X$ and $EWMA_{X^2}$		$\bar{X}$ chart, $EWMA_X$, and $EWMA_{X^2}$		$\bar{X}$ chart, $EWMA_X$, and $EWMA_{S^2}$		$EWMA_X$ and $EWMA_{X^2}$		$\bar{X}$ chart, $EWMA_X$, and $EWMA_{X^2}$	
			$\lambda_\mu = .10, \lambda_\sigma = .40$		$\lambda_\mu = .10$, $\lambda_\sigma = .40$	$\lambda_\mu = .18$, $\lambda_\sigma = .44$	$\lambda_\mu = .10, \lambda_\sigma = .40$		$\lambda_\mu = .40, \lambda_\sigma = .10$		$\lambda_\mu = .40$, $\lambda_\sigma = .10$	$\lambda_\mu = .43$, $\lambda_\sigma = .16$	$\lambda_\mu = .40, \lambda_\sigma = .10$	
	Size of shift		$\beta = 1$	$\beta = 5$			$\beta = 1$	$\beta = 5$	$\beta = 1$	$\beta = 5$			$\beta = 1$	$\beta = 5$
Performance measure	$\delta = \frac{\lvert\mu-\mu_0\rvert}{\sigma_0}$	$\psi = \frac{\sigma}{\sigma_0}$	Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9	Column 10	Column 11	Column 12
Normal ATS	0	1.00	1,481.4	1,481.4	1,481.4	1,481.4	1,481.4	1,481.3	1,481.5	1,481.6	1,481.7	1,481.4	1,481.5	1,481.6
Normal SSATS	.25	1.00	147.6	136.1	131.6	170.9	144.3	133.5	368.7	332.9	303.7	327.3	344.0	310.2
	.50	1.00	43.1	41.5	40.5	41.5	42.3	40.8	69.3	63.6	59.7	64.0	66.1	60.6
	.75	1.00	23.2	23.1	22.5	20.6	22.7	22.5	24.5	23.4	22.5	23.4	23.8	22.7
	1.00	1.00	14.8	15.3	15.0	13.2	14.4	14.7	12.8	12.6	12.4	12.5	12.5	12.3
	1.50	1.00	6.6	7.7	7.7	6.9	6.4	7.2	5.5	5.9	6.0	5.9	5.4	5.7
	2.00	1.00	3.2	3.9	4.2	4.0	3.1	3.6	3.0	3.3	3.8	3.6	2.9	3.3
	3.00	1.00	2.0	2.0	2.1	2.1	2.0	2.0	2.0	2.0	2.1	2.1	2.0	2.0
	4.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	5.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	7.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	10.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	15.00	1.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	0	1.20	172.4	168.7	147.1	150.3	159.1	148.7	108.3	104.2	88.3	99.9	95.0	89.2
	0	1.40	47.0	46.5	38.9	39.9	41.6	39.2	34.4	33.8	29.4	30.2	30.7	29.5
	0	1.60	21.4	21.4	18.1	18.4	19.1	18.2	18.5	18.3	16.4	16.1	16.9	16.4
	0	1.80	12.7	12.8	11.1	11.2	11.5	11.1	12.1	12.1	11.1	10.6	11.4	11.1
	0	2.00	8.8	8.9	7.8	7.9	8.1	7.9	8.9	8.9	8.3	7.9	8.4	8.3
	0	2.40	5.4	5.4	4.9	4.9	5.0	5.0	5.7	5.7	5.5	5.2	5.5	5.4
	0	3.00	3.5	3.6	3.3	3.3	3.4	3.3	3.8	3.8	3.7	3.5	3.7	3.7
	0	5.00	2.6	2.3	2.2	2.2	2.2	2.2	2.3	2.3	2.3	2.3	2.3	2.3
	0	7.00	2.1	2.1	2.1	2.1	2.1	2.1	2.1	2.1	2.1	2.1	2.1	2.1
	0	10.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
	0	15.00	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
Normal ATS	0	1.00	1,481.4	1,481.4	1,481.4	1,481.4	1,481.4	1,481.3	1,481.5	1,481.6	1,481.7	1,481.4	1,481.5	1,481.6
Laplace ATS	0	1.00	282.9	278.6	219.3	215.3	236.9	222.3	342.1	343.7	293.2	258.1	303.8	294.1
$t(4)$ ATS	0	1.00	238.0	235.1	193.1	190.0	204.5	195.2	294.1	296.6	260.4	229.8	266.7	261.0
$t(10)$ ATS	0	1.00	588.4	581.6	485.4	477.1	519.0	491.2	677.3	680.4	610.7	552.4	627.4	611.6
Gam(1, 1) ATS	0	1.00	185.0	182.3	143.8	141.7	152.8	145.2	229.1	232.1	202.0	175.6	205.2	201.3
Gam(2, 1) ATS	0	1.00	280.5	278.8	212.9	209.8	227.2	215.0	339.1	347.0	302.0	261.3	303.6	299.5
Gam(4, 1) ATS	0	1.00	434.7	434.7	327.8	323.1	350.7	330.9	511.5	524.5	461.7	402.6	461.1	456.5
Beta(3, 3) ATS	0	1.00	3,252.4	3,134.3	2,968.8	3,111.7	3,068.3	3,038.4	3,503.3	3,380.2	3,396.6	3,608.2	3,478.3	3,484.1
Beta(4, 4) ATS	0	1.00	2,930.1	2,925.6	2,876.4	2,987.8	2,810.7	2,899.8	2,965.3	2,916.1	3,012.9	3,207.9	2,998.0	3,061.2
Control chart parameters	h_X		3.3116	3.6627			3.2943	3.6500	3.2918	3.6540			3.2736	3.6380
	h_{S^2}													
	h_{EX}		3.0837	2.9829	2.9465	3.0640	3.0631	2.9656	3.2646	3.1877	3.1559	3.1613	3.2457	3.1691
	h_{ES^2}		4.8933	4.6790					3.6861	3.5428				
	h_{EX^2}				4.3993	4.4710	4.6242	4.4356			3.4020	3.6985	3.5605	3.4253

When setting up control charts to monitor a process, one question that arises is whether it is better to take small frequent samples, such as $n = 1$ and $d = 1.0$, or larger, less frequent samples, such as $n = 4$ and $d = 4.0$. Reynolds and Stoumbos (2004a, b) recently did an extensive investigation of this issue, and they concluded that the best choice of sample size n for EWMA or CUSUM chart combinations for monitoring μ and σ is $n = 1$. The results in Tables 1–3 for $n = 1$ and Tables 4–6 for $n = 4$ support and extend these conclusions to the control chart combinations with the added Shewhart limit. In particular, for EWMA chart combinations, $n = 1$ and $n = 4$ give roughly the same performance for small shifts, $n = 4$ is somewhat better for intermediate shifts, but $n = 1$ is much better for large shifts. There is little difference in robustness between the EWMA chart combinations in Tables 1–3 with $n = 1$ and the corresponding combinations in Tables 4–6 with $n = 4$.

7. TRANSIENT SHIFTS

Table 7 gives the probability of a signal by the control charts in Table 2 (where $n = 1$ and $d = 1.0$) when there is a transient shift of duration $l = 1.0$, 2.0, or 4.0 hours. The signal probabilities are the probabilities of a signal during the time that the transient shift is present or within four hours afterward.

If the only type of shift that can occur is a transient shift of very short duration ($l = 1.0$ hours), then the best way to monitor the process is to take samples of $n = 1$ every hour and use only a Shewhart X chart. Looking at the signal probabilities for $l = 1$ in Table 7 shows that the Shewhart X chart has the highest probability of a signal for all shifts in μ or σ. However, in most applications it would also be important to detect sustained shifts or transient shifts of longer duration, so using the X chart alone is certainly not the best option.

In Table 7, compare the EWMA$_X$ and EWMA$_{X^2}$ chart combination in columns 6 and 7 with the EWMA$_X$ and Shewhart X chart combination in columns 3–5. For shifts in μ when $l = 1.0$, the EWMA$_X$ and EWMA$_{X^2}$ chart combination is not as good as the EWMA$_X$ and Shewhart X chart combination when $\beta = 1$, but is better when $\beta > 1$. When $l = 2.0$ or 4.0, the EWMA$_X$ and EWMA$_{X^2}$ chart combination is better for all values of β. Adding the Shewhart X chart to the EWMA$_X$ and EWMA$_{X^2}$ chart combination increases the signal probability a little when $l = 1.0$, and decreases the signal probability slightly when $l = 2.0$ or 4.0.

We conclude that the EWMA$_X$ and EWMA$_{X^2}$ chart combination is not quite as effective as combinations involving the Shewhart X chart when the duration in the transient shift is very short ($l = 1.0$), but may be more effective when the duration is longer. This occurs because the Shewhart X chart uses the information in only the current observation, and thus is most effective when $l = 1.0$. When $l > 1.0$, the EWMA$_{X^2}$ chart accumulates information over several observations, and thus can be more effective than the Shewhart chart. Note that if detecting transient shifts is important, then the value of λ_σ used in the EWMA$_{X^2}$ chart should not be extremely small.

8. WHY IS THE EWMA$_{X^2}$ CHART SO EFFECTIVE FOR DETECTING LARGE MEAN SHIFTS?

When considering the performance of the different combinations of control charts, the question arises as to why the EWMA$_{X^2}$ chart, which is designed to detect variance increases, is so effective for detecting large mean shifts. We will try to provide some insight into this question. The fact that the sum of squared deviations from target can be expressed as the sum of independent chi-squared variables has been used previously in SPC applications of this statistic (see, e.g., Reynolds and Ghosh 1981), and this result gives

$$\sum_{i=1}^{n} \frac{(X_{ki} - \mu_0)^2}{n} = \frac{(n-1)S_k^2}{n} + (\bar{X}_k - \mu_0)^2$$
$$= \left[\chi^2(n-1, 0) + \chi^2\left(1, n\frac{(\mu - \mu_0)^2}{\sigma^2}\right)\right]\frac{\sigma^2}{n},$$

where $\chi^2(\nu, \eta)$ represents a noncentral chi-squared random variable with ν degrees of freedom and noncentrality parameter η. If σ remains constant and μ increases, then we see that the noncentrality parameter of the second chi-squared variable increases as the *square* of $(\mu - \mu_0)$, $(\mu - \mu_0)^2$, so a large change in μ will result in a very large noncentrality parameter.

Consider the specific situation in which $\mu_0 = 0$, $\sigma_0 = 1.0$, $n = 1$, and there is an upward mean shift of size $\delta = 4.0$. If the EWMA$_X$ chart is used alone with $\lambda_\mu = .02600$ and UCL .325, then the in-control ATS will be 1,481.6. If we assume for simplicity that the EWMA statistic is $E_{k-1}^X = 0$ immediately before the mean shift, then the expected value of E_k^X immediately after the shift will be $(1 - \lambda_\mu)(0) + \lambda_\mu E(X_k) = (.02600) \times (4.0) = .104$. Thus, in the first observation after the shift, the expected value of the EWMA statistic is not close to the UCL, and quick detection is unlikely. The steady-state probability that the EWMA$_X$ chart detects the shift in one observation is only .02, and the probability of detection within two observations is .14. These results are given in column 1 of Table 8.

Next, consider the EWMA$_{X^2}$ chart used alone with $\lambda_\sigma = .02600$, $UCL = 1.531$, and in-control ATS 1,481.6. Now, after the mean shift, $E(X_k^2) = (1.0 + \mu^2) = 17.0$. If we assume for simplicity that the EWMA statistic $E_{k-1}^{X^2}$ is at the reset value 1.0 immediately before the shift, then the expected value of $E_k^{X^2}$ immediately after the shift will be $E(X_k^2) = (.974)(1.0) + (.026)(17.0) = 1.416$, which is close to the UCL. The steady-state probability that the EWMA$_{X^2}$ chart detects the shift in one observation is .43, and the probability of detection within two observations is .91. These results are given in column 2 of Table 8. Thus we see that a large mean shift tends to move the EWMA$_{X^2}$ chart statistic close to the UCL, and the likelihood is high that a signal will occur very soon.

Columns 3 and 5 of Table 8 give results for the EWMA$_X$ chart with $\lambda_\mu = .11989$ and .50, and columns 4 and 6 give results for the EWMA$_{X^2}$ chart with $\lambda_\sigma = .11989$ and .50. The ability of the EWMA$_X$ chart to detect the shift improves as λ_μ increases, but even with the quite large value of $\lambda_\mu = .50$, the EWMA$_X$ chart has a lower probability of detecting the shift within one observation than the EWMA$_{X^2}$ chart with the lower value of $\lambda_\sigma = .11989$.

Table 7. Signal Probabilities for Transient Shifts in μ or σ for Matched Shewhart and EWMA Control Chart Combinations When $n = 1$, $d = 1.0$, and $(\lambda_\mu, \lambda_\sigma) = (.11989, .11989)$

Duration of shift	Size of shift: $\delta = \frac{\lvert\mu-\mu_0\rvert}{\sigma_0}$	Size of shift: $\psi = \frac{\sigma}{\sigma_0}$	X chart	$EWMA_X$ $\lambda_\mu = .11989$	X chart and $EWMA_X$, $\lambda_\mu = .11989$, $\beta = 1$	X chart and $EWMA_X$, $\lambda_\mu = .11989$, $\beta = 5$	X chart and $EWMA_X$, $\lambda_\mu = .11989$, $\beta = 10$	$EWMA_X$ and $EWMA_{X^2}$, $\lambda_\mu = .11989$, $\lambda_\sigma = .11989$	$EWMA_X$ and $EWMA_{X^2}$, $\lambda_\mu = .13110$, $\lambda_\sigma = .13110$	X chart, $EWMA_X$, and $EWMA_{X^2}$, $\lambda_\mu = .11989$, $\lambda_\sigma = .11989$, $\beta = 1$	X chart, $EWMA_X$, and $EWMA_{X^2}$, $\beta = 5$	X chart, $EWMA_X$, and $EWMA_{X^2}$, $\beta = 10$
			Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9	Column 10
$l = 1.00$	1.00	1.00	.01	.01	.01	.01	.01	.01	.01	.01	.01	.01
	2.00	1.00	.08	.02	.07	.05	.04	.05	.06	.07	.06	.05
	3.00	1.00	.35	.05	.29	.22	.18	.24	.24	.29	.24	.24
	4.00	1.00	.73	.12	.67	.57	.52	.59	.60	.67	.59	.59
	5.00	1.00	.95	.24	.92	.88	.85	.88	.89	.92	.88	.88
	7.00	1.00	1.00	.58	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0	1.50	.03	.01	.02	.02	.01	.02	.02	.02	.02	.02
	0	2.00	.09	.02	.08	.06	.06	.07	.07	.08	.07	.07
	0	3.00	.26	.07	.24	.21	.19	.22	.22	.24	.22	.22
	0	5.00	.50	.22	.48	.45	.43	.46	.46	.48	.46	.46
	0	7.00	.63	.37	.61	.59	.58	.59	.60	.61	.59	.59
	0	10.00	.73	.52	.72	.71	.70	.71	.71	.72	.71	.71
$l = 2.00$	1.00	1.00	.02	.02	.02	.02	.02	.02	.02	.02	.02	.02
	2.00	1.00	.16	.10	.15	.13	.12	.19	.19	.18	.19	.19
	3.00	1.00	.57	.34	.55	.48	.45	.65	.65	.64	.65	.65
	4.00	1.00	.92	.67	.91	.87	.84	.96	.96	.95	.96	.96
	5.00	1.00	1.00	.91	1.00	.99	.99	1.00	1.00	1.00	1.00	1.00
	7.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0	1.50	.05	.02	.04	.03	.03	.04	.04	.05	.04	.04
	0	2.00	.17	.05	.15	.12	.11	.15	.15	.16	.15	.15
	0	3.00	.45	.16	.42	.38	.36	.42	.43	.44	.43	.42
	0	5.00	.75	.42	.73	.70	.69	.73	.73	.74	.73	.73
	0	7.00	.86	.61	.85	.83	.82	.85	.85	.86	.85	.85
	0	10.00	.93	.77	.92	.91	.91	.92	.93	.93	.92	.92
$l = 4.00$	1.00	1.00	.04	.08	.07	.08	.08	.07	.08	.07	.07	.07
	2.00	1.00	.29	.53	.50	.53	.53	.57	.58	.55	.57	.57
	3.00	1.00	.82	.95	.94	.95	.95	.99	.98	.97	.98	.98
	4.00	1.00	.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	5.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	7.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0	1.50	.09	.04	.08	.06	.06	.10	.10	.10	.10	.10
	0	2.00	.31	.11	.29	.24	.22	.33	.34	.34	.33	.33
	0	3.00	.70	.32	.67	.63	.60	.72	.72	.73	.72	.72
	0	5.00	.94	.68	.93	.91	.91	.95	.95	.95	.95	.95
	0	7.00	.98	.85	.98	.97	.97	.98	.98	.98	.98	.98
	0	10.00	1.00	.95	.99	.99	.99	1.00	1.00	1.00	1.00	1.00

Table 8. Expected Values of EWMA Statistics After a Shift of $\delta = 4.0$, and Steady-State Probabilities of a Signal Within One and Two Observations After the Shift

Size of shift $\delta = \frac{|\mu - \mu_0|}{\sigma_0} = 4.0$	$EWMA_X$ $\lambda_\mu = .02600$ $UCL = .325$ Column 1	$EWMA_{X^2}$ $\lambda_\sigma = .02600$ $UCL = 1.531$ Column 2	$EWMA_X$ $\lambda_\mu = .11989$ $UCL = .814$ Column 3	$EWMA_{X^2}$ $\lambda_\sigma = .11989$ $UCL = 2.661$ Column 4	$EWMA_X$ $\lambda_\mu = .50$ $UCL = 1.956$ Column 5	$EWMA_{X^2}$ $\lambda_\sigma = .50$ $UCL = 6.535$ Column 6	X chart $UCL = 3.400$ Column 7
$(1-\lambda_\mu)(0) + \lambda_\mu E(X_k)$	.104		.480		2.000		4.000
$(1-\lambda_\sigma)(1.0) + \lambda_\sigma E(X_k^2)$		1.416		2.918		9.000	
P(signal within 1 observation)	.02	.43	.09	.63	.57	.72	.73
P(signal within 2 observations)	.14	.91	.64	.97	.98	.96	.93

Column 7 of Table 8 contains results for the Shewhart X chart, which is equivalent to the $EWMA_X$ chart with $\lambda_\mu = 1.00$ and also to the $EWMA_{X^2}$ chart with $\lambda_\sigma = 1.00$. Note that among all of the cases considered, the Shewhart X chart has the highest probability of detection within one observation, but not for detection within two observations. The $EWMA_{X^2}$ chart has a higher probability than the Shewhart X chart of detection within two observations when $\lambda_\sigma = .11989$, and a probability almost as high for the very small $\lambda_\sigma = .02600$.

9. CONCLUSIONS AND DISCUSSION

The results presented here show that the widespread recommendation to add Shewhart limits to the $EWMA_X$ chart does significantly improve the overall performance for detecting shifts in μ. However, it is important to monitor both μ and σ, and the picture is somewhat different in this case. The best EWMA chart for monitoring σ is the $EWMA_{X^2}$ chart. The $EWMA_{X^2}$ chart serves essentially the same function as the Shewhart limits for detecting large shifts in μ, while providing better detection of shifts in σ. So our answer to the question in the title—"Should EWMA and CUSUM Charts Be Used With Shewhart Limits?"—is that adding Shewhart limits is not essential for good overall performance, but it is essential to use the $EWMA_{X^2}$ chart with the $EWMA_X$ chart. If the $EWMA_{X^2}$ chart is used with the $EWMA_X$ chart, then this two-chart combination will provide good overall performance in detecting both small and large shifts in μ and σ, and the addition of the Shewhart chart to make a three-chart combination can be considered optional, depending on the user's preference.

In addition to our main conclusion that the $EWMA_{X^2}$ chart serves essentially the same function as the Shewhart limits, several other conclusions follow from the results presented here. We have shown that when $n > 1$, the control chart combinations that include the $EWMA_{X^2}$ chart are uniformly better for monitoring μ and σ than combinations that include the $EWMA_{S^2}$ chart. Thus the $EWMA_{X^2}$ chart should be used instead of the $EWMA_{S^2}$ chart for the best performance in detecting shifts.

After a signal by a control chart, it is important to be able to diagnose the type, size, and time of onset of the process-parameter shift. In today's computerized environment, the control charts used as diagnostic aids do not necessarily have to be the same control charts used to determine when to signal. For example, if the $EWMA_X$ and $EWMA_{X^2}$ chart combination is routinely plotted and used to determine when to signal, then additional control charts, such as the Shewhart $\bar{X}$ chart or the $EWMA_{S^2}$ chart, could be called up after a signal to be used as diagnostic aids whenever desired.

We have concluded that when the objective is to quickly detect large sustained or transient shifts, it is best to take small, frequent samples (i.e., use $n = 1$ and $d = 1.0$ rather than $n = 4$ and $d = 4.0$) and use a chart combination that includes a chart, such as the Shewhart X chart or the $EWMA_{X^2}$ chart, that reacts quickly to a single extreme observation. But such charts will necessarily be nonrobust to heavy-tailed distributions; the quick reaction to extreme observations will produce an increase in false alarms when the process is in control and the extreme observations are due only to the heavy-tailed distribution. Thus, adding the Shewhart X chart and/or the $EWMA_{X^2}$ chart to the $EWMA_X$ chart is in direct conflict with the desire to have a control chart combination that is robust to heavy-tailed distributions. Using a chart based on absolute deviations from target, as recommended earlier (see Stoumbos and Reynolds 2000; Reynolds and Stoumbos 2004a), can produce a robust control chart combination, but such a combination does not react as quickly to large sustained or transient shifts.

Although the numerical results and discussion here has been in terms of EWMA charts, the conclusions obtained apply directly to CUSUM charts. Numerical results for CUSUM chart combinations for monitoring μ and σ that include the Shewhart limits are available from the authors on request. (See also Reynolds and Stoumbos 2004a for numerical results showing that CUSUM charts based on sample means and squared deviations from target can be tuned to have very similar performance to EWMA charts based on sample means and squared deviations from target.)

ACKNOWLEDGMENTS

The authors thank the editor, associate editor, and referees for their comments and suggestions that greatly improved the presentation of the material in this article. Stoumbos' work was funded in part by the Law School Admission Council (LSAC) and by 2002 and 2003 Rutgers Business School Research Fellowships. The opinions and conclusions contained in this publication are those of the authors and do not necessarily reflect the position or policy of LSAC.

[Received November 2003. Revised March 2005.]

REFERENCES

Blacksell, S. D., Glefson, L. J., Lunt, R. A., and Chamnpood, C. (1994), "Use of Combined Shewhart–CUSUM Control Charts in Internal Quality Control of Enzyme-Linked Immunosorbent Assays for Typing of Foot and Mouth Disease Virus-Antigen," *Revue Scientifique et Technique de L'Office International des Epizooties*, 13, 687–699.

Borror, C. M., Montgomery, D. C., and Runger, G. C. (1999), "Robustness of the EWMA Control Chart to Non-Normality," *Journal of Quality Technology*, 31, 309–316.

Crowder, S. V. (1987), "A Simple Method for Studying Run-Length Distributions of Exponentially Weighted Moving Average Charts," *Technometrics*, 29, 401–407.

Crowder, S. V., and Hamilton, M. D. (1992), "An EWMA for Monitoring a Process Standard Deviation," *Journal of Quality Technology*, 24, 12–21.

Domangue, R., and Patch, S. C. (1991), "Some Omnibus Exponentially Weighted Moving Average Statistical Process Monitoring Schemes," *Technometrics*, 33, 299–313.

Gan, F. F. (1995), "Joint Monitoring of Process Mean and Variance Using Exponentially Weighted Moving Average Control Charts," *Technometrics*, 37, 446–453.

Hawkins, D. M., and Olwell, D. H. (1998), *Cumulative Sum Control Charts and Charting for Quality Improvement*, New York: Springer-Verlag.

Klein, M. (1996), "Composite Shewhart–EWMA Statistical Control Schemes," *IIE Transactions*, 28, 475–481.

Lucas, J. M. (1982), "Combined Shewhart–CUSUM Quality Control Schemes," *Journal of Quality Technology*, 14, 51–59.

Lucas, J. M., and Saccucci, M. S. (1990), "Exponentially Weighted Moving Average Control Schemes: Properties and Enhancements," *Technometrics*, 32, 1–12.

MacGregor, J. F., and Harris, T. J. (1993), "The Exponentially Weighted Moving Variance," *Journal of Quality Technology*, 25, 106–118.

Montgomery, D. C. (2005), *Introduction to Statistical Quality Control* (5th ed.), New York: Wiley.

Morais, M. C., and Pacheco, A. (2000), "On the Performance of Combined EWMA Schemes for μ and σ: A Markovian Approach," *Communications in Statistics, Part B—Simulation and Computation*, 29, 153–174.

Reynolds, M. R., Jr., and Ghosh, B. K. (1981), "Designing Control Charts for Means and Variances," in *35th Annual Quality Congress Transactions*, American Society for Quality Control, pp. 400–407.

Reynolds, M. R., Jr., and Kim, K. (2005), "Multivariate Monitoring the Process Mean Vector With Sequential Sampling," *Journal of Quality Technology*, 37, 149–162.

Reynolds, M. R., Jr., and Stoumbos, Z. G. (2001a), "Monitoring the Process Mean and Variance Using Individual Observations and Variable Sampling Intervals," *Journal of Quality Technology*, 33, 181–205.

——— (2001b), "Individuals Control Schemes for Monitoring the Mean and Variance of Processes Subject to Drifts," *Stochastic Analysis and Applications*, 19, 863–892.

——— (2004a), "Control Charts and the Efficient Allocation of Sampling Resources," *Technometrics*, 46, 200–214.

——— (2004b), "Should Observations Be Grouped for Effective Process Monitoring?" *Journal of Quality Technology*, 36, 343–366.

Roberts, S. W. (1959), "Control Charts Based on Geometric Moving Averages," *Technometrics*, 1, 239–250.

Ryan, T. P. (2000), *Statistical Methods for Quality Improvement* (2nd ed.), New York: Wiley.

Shamma, S. E., and Amin, R. W. (1993), "An EWMA Quality Control Procedure for Jointly Monitoring the Mean and Variance," *International Journal of Quality and Reliability Management*, 10, 58–67.

Stoumbos, Z. G., and Reynolds, M. R., Jr. (2000), "Robustness to Non-Normality and Autocorrelation of Individual Control Charts," *Journal of Statistical Computation and Simulation*, 66, 145–187.

Stoumbos, Z. G., Reynolds, M. R., Jr., and Woodall, W. H. (2003), "Control Chart Schemes for Monitoring the Mean and Variance of Processes Subject to Sustained Shifts and Drifts," in *Handbook of Statistics: Statistics in Industry*, 22, eds. C. R. Rao and R. Khattree, Amsterdam: Elsevier Science, pp. 553–571.

Stoumbos, Z. G., and Sullivan, J. H. (2002), "Robustness to Non-Normality of the Multivariate EWMA Control Chart," *Journal of Quality Technology*, 34, 260–276.

Westgard, J. O., Groth, T., Aronsson, T., and de Verdier, C. (1977), "Combined Shewhart–CUSUM Control Chart for Improved Quality Control in Clinical Chemistry," *Clinical Chemistry*, 23, 1881–1887.

Woodall, W. H. (2000),"Controversies and Contradictions in Statistical Process Control" (with discussion), *Journal of Quality Technology*, 32, 341–378.

Yashchin, E. (1993), "Statistical Control Schemes: Methods, Applications and Generalizations," *International Statistical Review*, 61, 41–66.